MAXIMUM PRINCIPLES FOR PARABOLIC SYSTEMS COUPLED
IN BOTH FIRST-ORDER AND ZERO-ORDER TERMS

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ABSTRACT. Some generalized maximum principles are established for linear second-order
parabolic systems in which both first-order and zero-order terms are coupled.

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1. INTRODUCTION.

Hile and Protter [2] proved that the Euclidean length of the solution vector
$u \in C^2(D) \cap C(\overline{D})$ of the second-order elliptic system

$$
\sum_{i,k=1}^{n} a_{ik}(x) \frac{\partial^2 u_k}{\partial x_i \partial x_k} + \sum_{j=1}^{m} b_{s,j}(x) \frac{\partial u_j}{\partial x_s} + \sum_{j=1}^{m} c_{s,j}(x) u_j = 0, \quad s = 1, \ldots, m,
$$

can be bounded by a constant times the maximum of its boundary values under a “small”
condition which requires that either the domain $D$ or the coefficients $b_{s,j}$ and $c_{s,j}$ are sufficiently
small. In this paper, we have established the same kind of maximum principle for the second-
order parabolic system

$$
\sum_{i,k=1}^{n} a_{ik}(x,t) \frac{\partial^2 u_k}{\partial x_i \partial x_k} - \frac{\partial u_k}{\partial t} + \sum_{j=1}^{m} b_{s,j}(x,t) \frac{\partial u_j}{\partial x_s} + \sum_{j=1}^{m} c_{s,j}(x,t) u_j = 0, 1 \leq s \leq m.
$$

Moreover, our parabolic version of the maximum principle holds without any “small” conditions.

When the coupling occurs only in the zero-order terms (i.e., in the case of $b_{s,j} = 0$ for all
$i, j, s$ except when $j = s$), the above systems are called weakly coupled systems. For weakly
coupled second-order parabolic systems, similar maximum principles have been obtained by Stys
[4] and Zhou [6]. Under different assumptions, different maximum principles in which the
components rather than the Euclidean length of the solution vector are bounded can be found in
Protter and Weinberger [3] and Dow [1]. In Weinberger’s paper [5], both kinds of maximum
principles have been reformulated and studied in terms of invariant sets.

2. MAIN RESULTS.

Consider a second-order parabolic operator with real coefficients,

$$
M \equiv \sum_{i,k=1}^{n} a_{ik}(x,t) \frac{\partial^2}{\partial x_i \partial x_k} - \frac{\partial}{\partial t} = a_{ij} = a_{ji},
$$
in a general bounded domain $\Omega$ in $\mathbb{R}^n \times \mathbb{R}_t$ ($n \geq 1$) with the boundary $\partial \Omega = \partial_p \Omega \cup \partial_q \Omega$. Here $\partial_p \Omega$
is the parabolic boundary of $\Omega$ and $\partial_q \Omega = \partial \Omega \setminus \partial_p \Omega$. We suppose that $\Omega \subset D \times (0,T)$ where $D$ is a
bounded domain in $\mathbb{R}^n$ and $0 < T < \infty$. The operator $M$ is assumed to be uniformly parabolic in $\Omega$; i.e., there is a constant $\delta > 0$ such that for all $(x, t) \in \Omega$ and all $(y_1, \cdots, y_n)$ in $C^n$ the inequality

$$
\sum_{i, k=1}^{n} a_{ik}(x, t) y_i y_k \geq \delta \sum_{i=1}^{n} |y_i|^2
$$

holds. The operator $M$ is the principal part of each equation in the second-order parabolic system

$$
Mu_s + \sum_{i=1}^{n} \sum_{j=1}^{m} b_{sj}(x, t) \frac{\partial u_j}{\partial x_i} + \sum_{j=1}^{m} c_{sj}(x, t) u_j = 0, \quad s = 1, 2, \cdots, m. \tag{2.2}
$$

We suppose that the complex-valued coefficients $b_{sj}$, $c_{sj}$ have the property that for all $\xi \in \mathbb{C}^m$ and all $(x, t) \in \Omega$,

$$
\sum_{r, s=1}^{m} \left[ c_{sr} + \bar{c}_{rs} + \frac{1}{2} \sum_{j=1}^{m} \sum_{k, i=1}^{n} A_{k} b_{sj} b_{kr} \right] \xi_{r} \xi_{s} \leq K ||\xi||^2, \text{for some } K > 0. \tag{2.3}
$$

Here $(A_{kj}) = (A_{jk})$ denotes the inverse matrix of $(a_{ik})$. A solution $u = (u_1, u_2, \cdots, u_m)$ is a complex-valued $C^{1}(\Omega \cup \partial \Omega) \cap C(\overline{\Omega})$ function which satisfies (2) in $\Omega$. Here $C^{1, k}(\Omega)$ is defined as the set of functions $f(x, t)$ having all $(x)$ space $(t)$ derivatives of order $\leq k$ and $t$ (time) derivatives of order $\leq h$ continuous in $\Omega$.

**THEOREM 1.** Assume conditions (1.1) and (1.3) hold. If $u$ is a solution of (2.2) and $\alpha$ is a positive $C^{1}(\Omega \cup \partial \Omega)$ function, then the product $\alpha |u|^{2} = \alpha \sum_{j=1}^{m} |u_j|^2$ cannot attain a positive maximum at any point in $\Omega \cup \partial \Omega$ where $\alpha$ satisfies

$$
\alpha^{-1}M\alpha - 2\alpha^{-2} \sum_{i, k=1}^{n} a_{ik} \frac{\partial \alpha}{\partial x_i} \frac{\partial \alpha}{\partial x_k} > K. \tag{2.4}
$$

**PROOF.** We set $p = |u|^2 = \sum_{s=1}^{m} |u_s|^2$ and find

$$
M(\alpha p) = pM\alpha + \alphaMp + 2 \sum_{i, k=1}^{n} a_{ik} \frac{\partial \alpha}{\partial x_i} \frac{\partial p}{\partial x_k}. \tag{2.5}
$$

At a point $(x, t) \in \Omega \cup \partial \Omega$ where $\alpha p$ attains a maximum, we have

$$
0 \leq \frac{\partial (\alpha p)}{\partial t}, \quad 0 = \frac{\partial (\alpha p)}{\partial x_k} = \alpha \frac{\partial p}{\partial x_k} + p \frac{\partial \alpha}{\partial x_k} \quad 1 \leq k \leq n,
$$

and (2.5) becomes

$$
M(\alpha p) = p \left[ M\alpha - 2\alpha^{-1} \sum_{i, k=1}^{n} a_{ik} \frac{\partial \alpha}{\partial x_i} \frac{\partial \alpha}{\partial x_k} \right] + \alphaMp. \tag{2.6}
$$

A direct computation yields

$$
Mp = \sum_{s=1}^{m} \left[ u_s, Mu_s + \bar{u}_s, Mu_s + 2 \sum_{i, k=1}^{n} a_{ik} \frac{\partial u_s}{\partial x_i} \frac{\partial \bar{u}_s}{\partial x_k} \right]
$$

$$
= \sum_{s=1}^{m} \left[ \sum_{j=1}^{m} b_{sj} \frac{\partial u_j}{\partial x_i} + \sum_{j=1}^{m} c_{sj} u_j \right] \frac{\partial \bar{u}_s}{\partial x_k} - \frac{1}{2} \sum_{q=1}^{m} \sum_{s=1}^{m} A_{q} b_{rs} b_{rq} \bar{u}_s \frac{\partial \bar{u}_s}{\partial x_k} \frac{\partial \bar{u}_s}{\partial x_k} + \sum_{s=1}^{m} \sum_{r, s=1}^{m} \sum_{k, q=1}^{m} A_{k} b_{sq} b_{rq} \bar{u}_s \frac{\partial \bar{u}_s}{\partial x_k} \frac{\partial \bar{u}_s}{\partial x_k}
$$

$$
\geq -K \sum_{s=1}^{m} |u_s|^2 = -Kp.
$$
Hence, from (2.6), we have

\[ M(\alpha p) \geq \alpha p \left[ \alpha^{-1} M_\alpha - 2\alpha^{-2} \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} \frac{\partial \alpha}{\partial x_i} \frac{\partial \alpha}{\partial x_k} - K \right] \]  

(2.7)

This inequality holds at any point in \( \Omega \cup \partial \Omega \) where \( \alpha p \) attains a maximum. Thus \( \alpha p \) cannot achieve a positive maximum at any point in \( \Omega \cup \partial \Omega \) where the quantity in brackets in (2.7) is positive. The theorem is established.

**Remark.** If for all \((x,t) \in \Omega, \)\n
\[ |c_{ij}| \leq K_{0}, \quad |b_{ij}| \leq K_{1}, \quad 1 \leq i \leq n, 1 \leq j, |s| \leq m, \text{ for some } K_{0}, K_{1} \in \mathbb{R}, \]  

(2.8)

then for any \( \xi \in \mathbb{C}^m, \)

\[ \sum_{r,s=1}^{m} \left[ c_{rs} + \delta_{rs} \right] \xi_r \xi_s = \sum_{j=1}^{m} \sum_{k=1}^{n} A_{kj} b_{ij} \xi_k \xi_j \xi_r \xi_s \leq \frac{m}{2} \sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{s=1}^{m} A_{ki} \left( \sum_{s=1}^{m} b_{ij} \xi_s \right) \left( \sum_{r=1}^{m} \xi_r \right) \leq 2mK_{0} \sum_{s=1}^{m} |\xi_s|^2 + \frac{1}{2} \sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{s=1}^{m} |b_{ij} \xi_s|^2 \leq 2mK_{0} + (2\delta)^{-1} nm^2 K_{1}^2 \]  

which is the condition (2.3) with \( K := 2mK_{0} + (2\delta)^{-1} nm^2 K_{1}^2. \) Hence, the single bound (2.3) in Theorem 1 can be replaced by the separate bounds (2.8) with \( K := 2mK_{0} + (2\delta)^{-1} nm^2 K_{1}^2. \)

Under the conditions (2.1) and (2.3) (or (2.1) and (2.8)), by choosing \( \alpha(x,t) = e^{-(K + \varepsilon)t}, \varepsilon > 0, \) the condition (2.4) will be satisfied. Hence from Theorem 1, we get the following maximum principle:

**Corollary 2 (Maximum Principle).** For any solution \( u \) of the system (2.2), the function

\[ |u(x,t)|^2 \exp\left[-(K + \varepsilon)t\right], \varepsilon > 0, \]

does not attain a positive maximum in \( \Omega \cup \partial \Omega, \) and

\[ \|u\|_{0,\Omega} \leq \exp(KT/2) \|u\|_{0,\partial \Omega}. \]  

(2.9)

Here \( K = (2\delta)^{-1} nm^2 K_{1}^2 + 2mK_{0} \) and \( \|u\|_{0,\Omega} := \sup_{(x,t) \in \Omega} |u(x,t)|. \)

**Remark.** Results similar to Theorem 1 and Corollary 2 for second-order elliptic systems were proven by Hile and Protter [2] (under a condition which is similar to (2.8)). But their maximum principle for elliptic systems only holds under the restriction that either the domain \( D \) is sufficiently small or the coefficients of the elliptic system are restricted sufficiently. Corollary 2 tells us that these restrictions can be lifted for parabolic systems.

**Corollary 3 (Uniqueness).** The system (2.2) with the initial-boundary condition

\[ u|_{\partial \Omega} = \varphi(x,t) \]

has at most one solution \( u \in C^{2,1}(\Omega \cup \partial \Omega) \cap C(\Omega). \)

Theorem 1 can be used to obtain bounds on the gradient of the \( C^{3,2} \) solution of the parabolic system (2.2), provided the coefficients are \( C^{1} \) and

\[ \|a_{ik}\|_{1,\Omega} \leq L_{2}, \|b_{ij}\|_{1,\Omega} \leq L_{1}, \|c_{ij}\|_{1,\Omega} \leq L_{0}, \text{ for some } L_{2}, L_{1}, L_{0} \in \mathbb{R}. \]  

(2.10)

Here \( \|f\|_{1,\Omega} := \|f\|_{0,\Omega} + \sum_{i=1}^{n} \|\frac{\partial f}{\partial x_i}\|_{0,\Omega} + \|\frac{\partial f}{\partial t}\|_{0,\Omega}. \)

We differentiate (2.2) with respect to \( x_{k} \) and \( t, \) and get \( m(n + 1) \) equations:
By combining (2.2), (2.11) and (2.12) we get a system (of the form (2.2)) consisting of \(m(n + 2)\) equations in the \(m(n + 2)\) unknowns \(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, s = 1, 2, \ldots, m, h = 1, 2, \ldots, n\).

**Theorem 4.** Let \(K = 2(n(n+2)m + \max\{L_1, L_2\}) + 2m(n+2)\max\{L_0, L_1\}\) and suppose that \(u\) is a \(C^{2,1}(\Omega \cup \partial_0 \Omega)\) solution of (2.2) and \(\alpha\) is a positive \(C^{2,1}(\Omega \cup \partial_0 \Omega)\) function. Then the product

\[
\alpha(x,t) \prod_{s=1}^{m} \left| u_{x_s} \right|^{2} + \left| \nabla u(x,t) \right|^{2} = \alpha(x,t) \sum_{s=1}^{m} \left[ \left| u_{x_s} \right|^{2} + \sum_{i=1}^{m} \left[ \left| \frac{\partial u_{x_s}}{\partial x_i} \right|^{2} + \left| \frac{\partial u_{x_s}}{\partial t} \right|^{2} \right] \right]
\]

cannot attain a positive maximum at any point in \(\Omega \cup \partial_0 \Omega\) where \(\alpha\) satisfies (2.4).

**Corollary 5.** Let \(K\) be the same number of Theorem 4. Then, for any \(C^{2,1}(\Omega \cup \partial_0 \Omega)\) solution \(u\) of the system (2.2), we have

\[
\| u \|_{\Omega} + \| \nabla u \|_{\Omega} \leq \exp(KT) \left( \| u \|_{\partial_0 \Omega} + \| \nabla u \|_{\partial_0 \Omega} \right)
\]
or equivalently,

\[
\| u \|_{1,\Omega} \leq \exp(KT/2) \cdot \| u \|_{1,\partial \Omega}
\]

**Remark.** Under the condition that either \((c_{x_s})_{m \times m}\) is a constant matrix or \((c_{x_s})_{m \times m}\) is invertible for all \((x,t) \in \Omega\), the unknowns \(u, s = 1, \ldots, m\), can be eliminated from the system (2.2), (2.11), (2.12), and then a system of \(m(n+1)\) equations in the gradient of \(u\) yields a maximum principle for \(\alpha \left| \nabla u \right|^{2}\).

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**References**