CR-HYPERSURFACES OF COMPLEX PROJECTIVE SPACE

M.A. BASHIR
Mathematics Department
College of Science
King Saud University
P. O. Box 2455
Riyadh 11451
Saudi Arabia

(Received December 17, 1991)

ABSTRACT. We consider compact n-dimensional minimal foliate CR-real submanifolds of a complex projective space. We show that these submanifolds are great circles on a 2-dimensional sphere provided that the square of the length of the second fundamental form is less than or equal to n – 1.

KEY WORDS AND PHRASES. Kaehler manifold, CR-submanifold, mixed foliate, hypersurfaces of complex projective space.

1991 AMS SUBJECT CLASSIFICATION CODES. Primary 53C40. Secondary 53C55.

1. INTRODUCTION.

CR-submanifolds of a Kaehlerian manifold have been defined by A. Bejancu [1]. These manifolds have then been studied by several authors. Among these are B.Y. Chen [2],[3], K. Yano, M. Kon, K. Sekigawa, and A. Ross [4].

In particular CR-submanifolds isometrically immersed in complex projective space have been considered by K. Yano and M. Kon [6]. They studied CR-submanifolds isometrically immersed in complex projective space with geometric properties such as semi-flat normal connection or parallel mean curvature. In this paper we consider minimal proper CR-hypersurfaces of a complex projective space. For such submanifolds we have obtained the following:

THEOREM 1. Let M be a compact n-dimensional minimal foliate CR-real hypersurface of a complex projective space. If the square of the length of the second fundamental form is \( \leq (n-1) \), then M is a totally real submanifold of dimension 1. In fact M is a great circle on \( S^2 \).

2. PRELIMINARIES.

A submanifold M of a Kaehler manifold is called a CR-submanifold if there is a differentiable distribution \( D:z \rightarrow D \subseteq \mathbb{T}M \) on M satisfying the following conditions:

(a) \( D \) is holomorphic i.e., \( JD = D \) for each \( z \in M \), where \( J \) is the almost complex structure.
(b) The complementary orthogonal distribution \( D:z \rightarrow D \subseteq \mathbb{T}M \) is totally real i.e., \( J\mathbb{D} \subseteq \mathbb{T}_zM \) where \( \mathbb{T}_zM \) is the normal bundle. If \( \dim D = 0 \) (respectively, \( \dim D = 0 \)), M is called a complex (respectively totally real) submanifold. A CR-submanifold is said to be proper if it is neither complex nor totally real. The normal bundle \( T_zM \) splits as \( T_zM = J\mathbb{D} \oplus \mu \), where \( \mu \) is invariant sub-bundle of \( T_zM \) under \( J \).
Now let $\bar{M}$ be the complex projective space, which is a Kaehler manifold with constant holomorphic sectional curvature 4. Let $g$ be the Hermitian metric tensor field of $\bar{M}$. Suppose that $M$ is an $n$-dimensional CR-hypersurface of $\bar{M}$. We denote by the same $g$ the Riemannian metric tensor field induced on $M$ from that of $\bar{M}$. Let $\nabla, \nabla, \nabla$ be the Riemannian connections on $M$, $\bar{M}$ and the normal bundle respectively. Then we have Gauss formula and Weingarten formula;

$$\nabla_X Y = \nabla_X Y + h(X,Y) \quad (2.1)$$

$$\nabla_X N = -A_N X, \quad N \in T_M \quad (2.2)$$

where $h(X,Y)$ and $A_N X$ are the second fundamental forms which are related by

$$g(h(X,Y),N) = g(A_N X, Y) \quad (2.3)$$

where $X$ and $Y$ are vector fields on $M$.

We also have the following Gauss equation

$$R(X,Y;Z,W) = g(Y,Z)g(X,W) - g(X,Z)g(Y,W) + g(JY,Z)g(JX,W) - g(JX,Z)g(JY,W)$$

$$+ 2g(X,JY)g(JZ,W) + h(Y,Z)h(X,W) - h(X,Z)h(Y,W) \quad (2.4)$$

where $R(X,Y;Z,W)$ is the Riemannian curvature tensor of type (0,4).

Let $H = \frac{1}{n} (\text{trace } h)$ be the mean curvature vector. Then $M$ is said to be minimal if $H = 0$.

A CR-submanifold is said to be mixed foliate if

(a) the holomorphic distribution $D$ is integrable.

(b) $h(X,\xi) = 0$ for $X \in D$ and $\xi \in \overline{D}$.

For mixed foliate submanifolds of a complex space form $\bar{M}(c)$ (i.e., a Kaehler manifold of constant holomorphic sectional curvature $c$), the following result is well known

**THEOREM 2.** [3] If $M$ is a mixed foliate proper CR-submanifold of a complex space form $\bar{M}(c)$, then we have $c \leq 0$.

3. **CR-HYPERSURFACES OF A COMPLEX PROJECTIVE SPACE.**

We consider an $n$-dimensional proper CR-hypersurface $M$ of a complex projective space $\bar{M}$. Then it follows that $\dim D = 1$. Now assume that $M$ is minimal and the holomorphic distribution $D$ is integrable. If $(\xi_i), i = 1, \ldots, 2p$ is an orthonormal basis for $D$, where $2p = \dim D$, then $\sum_{i=1}^{2p} h(\xi_i, \xi_i) = 0$. Since $M$ is minimal we get $h(\xi, \xi) = 0$ for $\xi$ a unit vector in $D$. Note that $\nabla_X \xi \in D$. Then using the equation $\nabla_X J\xi = J\nabla_X \xi$ and equations (2.1) and (2.2) we have for $X \in D$

$$\nabla_X \xi = JAX - h(X,\xi) \quad (3.1)$$

Also the equation $\nabla_{\xi} J\xi = J\nabla_{\xi} \xi$ with $h(\xi, \xi) = 0$ and equations (2.1) and (2.2) yields

$$\nabla_{\xi} \xi = JA\xi \quad (3.2)$$

Let $(\xi_i), i = 1, \ldots, n$ be an orthonormal basis for $M$, where $\xi_i = \bar{\xi}_i$ for $i = 1, \ldots, 2p$ and $\xi_n = \xi$. $n = 2p + 1$. Since $A$ is symmetric and $J$ is skew symmetric we get

$$g(JA\xi_i, \xi_i) = -g(JAJ\xi_i, J\xi_i) \quad (3.3)$$

Then using (3.1), (3.2), and (3.3) we compute

$$\text{div } \xi = \sum_{i=1}^{2p} g(\nabla_{\xi_i} \xi_i, \xi_i) + \sum_{i=1}^{2p} g(\nabla_{\xi_i} \xi_i, \bar{\xi}_i) = \sum_{i=1}^{p} [g(JA\xi_i, \xi_i) + g(JAJ\xi_i, J\xi_i)] = 0 \quad (3.4)$$
For any vector field $X$ on $M$ we have [5]

$$\text{div}(\nabla_X X) - \text{div}(\text{div}X)X = S(X, X) + \frac{1}{2} |L_X g|^2 - |\nabla X|^2 - (\text{div}X)^2$$  \hspace{1cm} (3.5)$$

where $S$ is the Ricci tensor and $L_X g$ is the Lie differentiation with respect to a vector field $X$, defined by

$$(L_X g)(Y, Z) = g(\nabla_X Y, Z) + g(\nabla_X Z, Y)$$

Using (3.4) in (3.5) with $X = \xi$ we get

$$\text{div}(\nabla_\xi \xi) = S(\xi, \xi) + \frac{1}{2} |L_\xi g|^2 - |\nabla \xi|^2$$  \hspace{1cm} (3.6)$$

From Gauss equation (2.4) and the fact that $h(\xi, \xi) = 0$ we have

$$S(\xi, \xi) = (n-1)g(\xi, \xi) - \sum_{i=1}^{\text{dim} M} g(h(\xi, \xi), h(\xi, \xi)) = (n-1) - \sum_{i=1}^{\text{dim} M} g(h(\xi, \xi), J\xi)g(h(\xi, \xi), J\xi)$$

Using (3.1) and (3.2) we also have

$$\text{tr}A^2 = g(\xi, \xi) = g(\xi, \xi) = g(\xi, \xi) = g(\xi, \xi)$$

Using (3.6), (3.7), and (3.8) we obtain

$$\text{div}(\nabla_\xi \xi) = (n-1) - \text{tr}A^2 + \frac{1}{2} |L_\xi g|^2$$  \hspace{1cm} (3.9)$$

PROOF. Using equation (3.9) and the assumption that $M$ is compact we have

$$2 \int_M [(n-1) - \text{tr}A^2] dv = - \int_M |L_\xi g|^2 dv$$  \hspace{1cm} (3.10)$$

From the hypothesis of Theorem and equation (3.10), we have $|L_\xi g| = 0$. Hence

Using equation (3.2) in the above equation we get $h(X, \xi) = 0$ i.e., $M$ is mixed foliate. Since the holomorphic sectional curvature $c$ of the complex projective space $\hat{M}$ equals 4, then by theorem (2) $M$ cannot be proper mixed foliate. Therefore $M$ is either totally real or holomorphic. But since $\text{dim} \hat{M} = 1$, $M$ cannot be holomorphic. Therefore $M$ is totally real. Since $M$ is a hypersurface this implies that $\text{dim} M = 1$ and $\text{dim} \hat{M} = 2$. Now using the assumption that $\text{tr}A^2 \leq n-1$ and $\text{dim} M = 1$ we have $\text{tr}A^2 = 0$ i.e., $M$ is totally geodesic. Since $\text{dim} \hat{M} = 2$ i.e., $\hat{M}$ is $S^2(= CP)$, then $M$ totally geodesic implies that $M$ is a great circle $S^1$ on $S^2$.

NOTE: It has been pointed out to us that the result in this theorem might be in conflict with Proposition 2.3 of Maeda, Y., "On real hypersurfaces of a complex projective space," J. Math. Soc. Japan, Vol. 28, No. 3 (1976), 529-540. We could not detect any mistakes in our proof, but we shall investigate this point later.
REFERENCES