GENERALIZED COMMON FIXED POINT THEOREMS FOR
A SEQUENCE OF FUZZY MAPPINGS

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ABSTRACT. We obtain generalized common fixed point theorems for a sequence of fuzzy mappings, which is a generalization of the result of Lee and Cho [6].

KEY WORDS AND PHRASES. Fuzzy set, fuzzy mapping, upper semi-continuous, common fixed point.

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1. INTRODUCTION. Heilpern [3] first introduced the concept of fuzzy mappings and proved a fixed point theorem for fuzzy contraction mappings, which is a fuzzy analogue of the fixed point theorems for multi-valued mappings ([2], [4], [9]) and the well-known Banach fixed point theorem. Bose and Sahani [1], in their first theorem, extended Heilpern's result for a pair of generalized fuzzy contraction mappings. They also, in their second theorem, proved a fixed point theorem for non-expansive fuzzy mappings on a compact star-shaped subset of a Banach space. Lee and Cho [5] proved a fixed point theorem for a contractive-type fuzzy mapping which is an extension of the result of Heilpern [3]. Also, they [6] obtained common fixed point theorems for a sequence of fuzzy mappings which are generalizations of their result in [5]. Lee et al. [7] obtained a common fixed point theorem for a sequence of fuzzy mappings satisfying certain conditions, which is a generalization of the second theorem of Bose and Sahani. They also showed common fixed point theorems for a pair of fuzzy mappings in [8], which is an extension of the first theorem of Bose and Sahani [1].

In this paper, we prove generalized common fixed point theorems for a sequence of fuzzy mappings satisfying certain conditions which are generalizations of the result of Lee and Cho [6].

2. PRELIMINARIES.

Let \((X,d)\) be a linear metric linear space. A fuzzy set \(A\) in \(X\) is a function from \(X\) into \([0,1]\). If \(x \in X\), the function value \(A(x)\) is called the grade of membership of \(x\) in \(A\). The \(\alpha\)-level set of \(A\), denote by \(A_\alpha\), is defined by

\[ A_\alpha = \{ x : A(x) \geq \alpha \} \quad \text{if} \quad \alpha \in (0,1], \quad A_0 = \{ x : A(x) > 0 \}. \]
where \( \overline{B} \) denotes the closure of the nonfuzzy set of \( B \).

Let \( W(X) \) be the collection of all the fuzzy sets \( A \) in \( X \) such that \( A_\alpha \) is compact and convex for each \( \alpha \in [0,1] \), and \( \sup_{x \in X} A(x) = 1 \). For \( A, B \in W(X) \), \( A \subseteq B \) means \( A(x) \leq B(x) \) for each \( x \in X \).

**DEFINITION 2.1.** Let \( A, B \in W(X) \) and \( \alpha \in [0,1] \). Then we define

\[
P_\alpha(A, B) = \inf_{x \in A_\alpha, y \in B_\alpha} d(x, y), \quad P(A, B) = \sup_{\alpha} P_\alpha(A, B)
\]

and

\[
D(A, B) = \sup_{\alpha} d_H(A_\alpha, B_\alpha),
\]

where \( d_H \) is the Hausdorff metric induced by the metric \( d \). We note that \( P_\alpha \) is a nondecreasing function of \( \alpha \) and \( D \) is a metric on \( W(X) \).

**DEFINITION 2.2.** Let \( X \) be an arbitrary set and \( Y \) be any linear metric space. \( F \) is called a fuzzy mapping if and only if \( F \) is a mapping from the set \( X \) into \( W(Y) \).

In the following section, we will use the following lemmas.

**LEMMA 2.1** [5]. Let \( (X, d) \) be a complete linear metric space, \( F \) a fuzzy mapping from \( X \) into \( W(X) \) and \( x_0 \in X \), then there exists \( x_1 \in X \) such that \( \{x_1\} \subseteq F(x_0) \).

**LEMMA 2.2** [8]. Let \( A, B \in W(X) \). Then for each \( \{z\} \subseteq A \) there exists \( \{y\} \subseteq B \) such that \( D(\{z\}, \{y\}) \leq D(A, B) \).

We can easily prove the following lemma.

**LEMMA 2.3.** Let \( x \in X \) and \( B \in W(X) \). If \( \{y\} \subseteq B \), then \( P(\{x\}, B) \leq d(x, y) \).

3. **COMMON FIXED POINTS THEOREMS FOR A SEQUENCE OF FUZZY MAPPINGS.**

**THEOREM 3.1.** Let \( g \) be a non-expansive mapping from a complete linear metric space \( (X, d) \) into itself. If \( (F_i)_{i=1}^\infty \) is a sequence of fuzzy mappings from \( X \) into \( W(X) \) satisfying the following condition: For each pair of fuzzy mappings, \( F_i, F_j \) and for any \( x \in X \), \( \{u_i\} \subseteq F_i(x) \), there exists \( \{v_j\} \subseteq F_j(y) \) for all \( y \in X \) such that

\[
D(\{u_i\}, \{v_j\}) \leq a_1d(g(x), g(u_i)) + a_2d(g(y), g(v_j)) + a_3d(g(x), g(y)),
\]

where \( a_1, a_2, a_3, a_4, a_5 \) are nonnegative real numbers, \( a_1 + a_2 + a_3 + a_4 + a_5 < 1 \) and \( a_3 \geq a_4 \). Then there exists \( p \in X \) such that \( \{p\} \subseteq \bigcap_{i=1}^\infty F_i(p) \).

**PROOF.** Let \( x_0 \in X \). Then we can choose \( x_1 \in X \) such that \( \{x_1\} \subseteq F_1(x_0) \) by Lemma 2.1. By our assumptions, there exists \( x_2 \in X \) such that \( \{x_2\} \subseteq F_2(x_1) \) and

\[
D(\{x_1\}, \{x_2\}) \leq a_1d(g(x_0), g(x_1)) + a_2d(g(x_1), g(x_2)) + a_3d(g(x_1), g(x_1))
\]

\[
+ a_4d(g(x_0), g(x_2)) + a_5d(g(x_0), g(x_1))
\]

\[
\leq a_1d(x_0, x_1) + a_2d(x_1, x_2) + a_3d(x_1, x_1) + a_4d(x_0, x_2) + a_5d(x_0, x_1).
\]

Again we can find \( x_3 \in X \) such that \( \{x_3\} \subseteq F_3(x_2) \) and

\[
D(\{x_2\}, \{x_3\}) \leq a_1d(x_2, x_1) + a_2d(x_2, x_3) + a_3d(x_2, x_2) + a_4d(x_1, x_3) + a_5d(x_1, x_2).
\]

Inductively, we obtain a sequence \( (x_n) \) in \( X \) such that \( \{x_{n+1}\} \subseteq F_{n+1}(x_n) \) and

\[
D(\{x_n\}, \{x_{n+1}\}) \leq a_1d(x_{n-1}, x_n) + a_2d(x_n, x_{n+1}) + a_3d(x_{n-1}, x_n)
\]

\[
+ a_4d(x_{n-1}, x_{n+1}) + a_5d(x_{n-1}, x_n).
\](3.1)

Since \( D(\{x_n\}, \{x_{n+1}\}) = d(x_n, x_{n+1}) \), by (3.1)

\[
d(x_n, x_{n+1}) \leq a_1d(x_{n-1}, x_n) + a_2d(x_n, x_{n+1})
\]

\[
+ a_4d(x_{n-1}, x_n) + a_4d(x_n, x_{n+1}) + a_5d(x_{n-1}, x_n).
\]
Hence
\[ d(x_n, x_{n+1}) \leq [(a_1 + a_4 + a_5)/(1 - a_2 - a_4)]d(x_{n-1}, x_n). \]

Let \( r = (a_1 + a_4 + a_5)/(1 - a_2 - a_4). \) Since \( a_3 \geq a_4, \) \( 0 < r < 1. \) Moreover, we have \( d(x_n, x_{n+1}) \leq r^nd(x_0, x_1). \) We can easily show that \( (x_n)_{n=1}^{\infty} \) is a Cauchy sequence in \( X. \) Since \( X \) is complete, there exists \( p \in X \) such that \( \lim_{n \to \infty} x_n = p. \) Let \( F_m \) be an arbitrary member of \( (F_i)_{i=1}^{\infty}. \) Since \( \{x_n\} \subset F_n(x_{n-1}) \) for all \( n, \) there exists \( v_n \in X \) such that \( \{v_n\} \subset F_m(p) \) for all \( n \) and
\[ D(\{x_n\}, \{v_n\}) \leq a_1d(x_{n-1}, x_n) + a_2d(p, v_n) + a_3d(p, x_n) + a_4d(x_{n-1}, v_n) + a_5d(x_{n-1}, p). \] (3.2)

From (3.2), we have
\[ d(x_n, v_n) \leq a_1d(x_{n-1}, x_n) + a_2d(p, x_n) + a_3d(x_n, v_n) + a_4d(x_{n-1}, v_n) + a_5d(x_{n-1}, p) \]
Thus we have
\[ (1 - a_2 - a_4)d(x_n, v_n) \leq a_1d(x_{n-1}, x_n) + a_2d(p, x_n) + a_3d(x_n, v_n) + a_4d(x_{n-1}, x_n) + a_5d(x_{n-1}, p). \]
Since \( x_n \to p \) as \( n \to \infty, \) \( (1 - a_2 - a_4)d(x_n, v_n) \to 0 \) as \( n \to \infty. \) Hence \( d(x_n, v_n) \to 0 \) as \( n \to \infty. \) Since \( d(p, v_n) \leq d(p, x_n) + d(x_n, v_n), \) \( v_n \to p \) as \( n \to \infty. \) Since \( F_m(p) \in W(X), \) \( F_m(p) \) is upper semi-continuous and thus
\[ \lim_{n \to \infty} \sup \{F_m(p)(v_n)\} \leq \{F_m(p)(p)\}. \]
Since \( \{v_n\} \subset F_m(p) \) for all \( n, \) \( \{F_m(p)(p)\} = 1. \) Hence \( \{p\} \subset F_m(p). \) Since \( F_m \) is arbitrary, \( \{p\} \subset \bigcap_{i=1}^{\infty} F_i(p). \)

Putting \( g(x) = x, \) we get the following corollary from Theorem 3.1.

**COROLLARY 3.1.** Let \((X, d)\) be a complete linear metric space. If \((F_i)_{i=1}^{\infty} \) is a sequence of fuzzy mappings from \( X \) into \( W(X) \) satisfying the following condition \((\ast)\): For each pair of fuzzy mapping \( F_i, F_j \) and for any \( x \in X, \{u_j\} \subset F_i(x), \) there exists \( \{v_i\} \subset F_j(y) \) for all \( y \in X \) such that
\[ D(\{u_j\}, \{v_i\}) \leq a_1d(x, u_j) + a_2d(y, v_i) + a_3d(y, u_j) + a_4d(x, v_i) + a_5d(x, y), \]
where \( a_1, a_2, a_3, a_4, a_5 \) are nonnegative real numbers, \( a_1 + a_2 + a_3 + a_4 + a_5 < 1 \) and \( a_3 \geq a_4. \) Then there exists \( p \in X \) such that \( \{p\} \subset \bigcap_{i=1}^{\infty} F_i(p). \)

By Lemmas 2.2 and 2.3, we can obtain the following corollary from Corollary 3.1.

**COROLLARY 3.2.** Let \((X, d)\) be a complete linear metric space and let \((F_i)_{i=1}^{\infty} \) be a sequence of fuzzy mappings from \( X \) into \( W(X) \) satisfying the following condition \((\ast\ast)\): For each pair of fuzzy mappings \( F_i, F_j, \)
\[ D(F_i(x), F_j(y)) \leq a_1P(x, F_i(x)) + a_2P(y, F_j(y)) + a_3P(y, F_i(x)) + a_4P(x, F_j(y)) + a_5d(x, y), \]
for all \( x, y \in X, \) where \( a_1, a_2, a_3, a_4, a_5 \) are nonnegative real numbers, \( a_1 + a_2 + a_3 + a_4 + a_5 < 1 \) and \( a_3 \geq a_4. \) Then there exists \( p \in X \) such that \( \{p\} \subset \bigcap_{i=1}^{\infty} F_i(p). \)

The following example shows that the condition \((\ast)\) in Corollary 3.1 does not imply the condition \((\ast\ast)\) in Corollary 3.2.

**EXAMPLE 3.1.** Let \((F_i)_{i=1}^{\infty} \) be a sequence of fuzzy mappings from \([0, \infty)\) into \( W([0, \infty)), \)
where \( F_i(x); [0, \infty) \to [0, 1] \) is defined as follows
\[
\begin{align*}
&\text{if } x = 0, \quad [F_i(x)](z) = \begin{cases} 
1, & z = 0 \\
0, & z \neq 0,
\end{cases} \\
&\text{otherwise, } [F_i(x)](z) = \begin{cases} 
1, & 0 \leq z \leq x/2 \\
1/2, & x/2 < z \leq ix \\
0, & z > ix.
\end{cases}
\end{align*}
\]
Then the sequence \((F_n)_{n=1}^\infty\) satisfies the condition \((*)\) when \(a_1 = a_2 = a_3 = a_4 = 0\), but does not satisfy the condition \((**\))

Putting \(a_1 = a_2 = a_3 = a_4 = 0\), we get the following corollary from Theorem 3.1.

**COROLLARY 3.3** [6]. Let \(g\) be a non-expansive mapping from a complete linear metric space \((X, d)\) into itself and \((F_n)_{n=1}^\infty\) a sequence of fuzzy mappings from \(X\) into \(W(X)\) satisfying the following condition: There exists a constant \(k\) with \(0 < k < 1\) such that for each pair of fuzzy mappings \(F_n, F_j\) and for any \(x \in X, \{u_x\} \subseteq F_n(x)\), there exists \(\{v_y\} \subseteq F_j(y)\) for all \(y \in X\) such that

\[
D(\{u_x\}, \{v_y\}) \leq kd(g(x), g(y)).
\]

Then there exists \(p \in X\) such that \(\{p\} \subseteq \bigcap_{n=1}^\infty F_n(p)\).

By Lemma 2.2, we get the following corollary from Corollary 3.3.

**COROLLARY 3.4** [6]. Let \(g\) be a non-expansive mapping from a complete linear metric space \((X, d)\) into itself and \((F_n)_{n=1}^\infty\) a sequence of fuzzy mappings from \(X\) into \(W(X)\) satisfying the following condition: There exists a constant \(k\) with \(0 < k < 1\) such that for each pair of fuzzy mappings \(F_n, F_j\),

\[
D(F_n(x), F_j(y)) \leq kd(g(x), g(y)) \quad \text{for all } x, y \in X,
\]

Then there exists \(p \in X\) such that \(\{p\} \subseteq \bigcap_{n=1}^\infty F_n(p)\).

Putting \(g(x) = x\), we get the following corollary from Corollary 3.4.

**COROLLARY 3.5** [6]. Let \((X, d)\) be a complete linear metric space and \((F_n)_{n=1}^\infty\) be a sequence of fuzzy mappings from \(X\) into \(W(X)\) satisfying the following condition. There exists a constant \(k\) with \(0 < k < 1\) such that for each pair of fuzzy mappings \(F_n, F_j\),

\[
D(F_n(x), F_j(y)) \leq kd(x, y) \quad \text{for all } x, y \in X
\]

Then there exists \(p \in X\) such that \(\{p\} \subseteq \bigcap_{n=1}^\infty F_n(p)\).

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