CLASSIFICATION OF SOLUTIONS OF DELAY DIFFERENCE EQUATIONS

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ABSTRACT. In this paper we study the classification of solutions of delay difference equation

\[ \begin{cases}
\Delta^2 y_n = P_n y_{n-m} \\
y_n = A_n \text{ for } n = N - (m+1), \ldots, N-1
\end{cases} \]

where \( A_n, n = N - (m+1), \ldots, N-1 \) are given, \( m \) is a nonnegative integer.

KEY WORDS AND PHRASES. Delay difference equations, oscillation, bounded solutions.

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1. INTRODUCTION. The problem of oscillation and nonoscillation of solutions of delay difference equations has been receiving a lot of attention for the last few years. Erbe and Zhang ([1]-[3]), Lalli, Zhang and Zhao ([8], [9]), Ladas, Philos and Sficas ([6], [7]), have done some extensive works on this topic. A survey on the oscillation of delay difference equations could be found in the monograph by Gyori and Ladas [5].

In this paper we consider the second order delay difference equations of the form:

\[ \Delta^2 y_n = P_n y_{n-m} \] (1.1)

where \( \Delta \) denotes the forward difference operator: \( \Delta y_n = y_{n+1} - y_n, m \) is a nonnegative integer.

By a solution of equation (1.1) we mean a sequence \( \{y_n\} \) which is defined for \( n \geq N - (m+1) \) and which satisfies equations (1.1) for all \( n \geq N \). Clearly if

\[ y_n = A_n, \text{ for } n = N - (m+1), N - m, \ldots, N \] (1.2)

are given, then equation (1.1) has a unique solution satisfying the initial conditions (1.2), where \( N \) is an initial point.

A nontrivial solution \( \{y_n\} \) of equation (1.1) is said to be oscillatory if for every \( N > 0 \) there exists an \( n \geq N \) such that \( y_n y_{n+1} \leq 0 \). Otherwise it is called nonoscillatory.

Set \( E_N = \{N - (m+1), N - m, \ldots, N - 1\} \), if

\[ y_n = A_n, n \in E_N \] (1.3)
are given, then the solutions depend on the parameter $y_N = \xi$. We are concerning with the classification of solutions of equation (1.1) with (1.3).

2. MAIN RESULTS.

We always assume that $P_n \geq 0$ and $P_n$ does not identically equal to zero in equation (1.1). We denote $S$ the set of all solutions of (1.1). Since $P_n \geq 0$, it is easy to see that

$$S = S^+ \cup S^- \cup S^k \cup S^{-k} \cup S^0 \cup S^\sim$$

where

$$S^+ = \{\{y_n\} \in S: \lim_{n \to \infty} y_n = +\infty\}$$
$$S^- = \{\{y_n\} \in S: \lim_{n \to \infty} y_n = -\infty\}$$
$$S^k = \{\{y_n\} \in S: 0 < \lim_{n \to \infty} y_n < +\infty\}$$
$$S^{-k} = \{\{y_n\} \in S: 0 > \lim_{n \to \infty} y_n > -\infty\}$$
$$S^0 = \{\{y_n\} \in S: y_n \text{ nontrivial, } \lim_{n \to \infty} y_n = 0 \text{ monotonically}\}$$
$$S^\sim = \{\{y_n\} \in S: y_n \text{ is oscillatory}\}$$

**LEMMA 2.1** If

$$y_i \geq 0 \text{ on } E_N, \ y_N > y_{N-1}$$

then $y_n \in S^+$. If

$$y_i \leq 0 \text{ on } E_N, \ y_N < y_{N-1}$$

than $y_n \in S^-$. 

**PROOF.** From (1.1), we have

$$\sum_{i=0}^{N-1} P_i y_{i-m} = \sum_{i=0}^{N-1} P_i y_{i-m}$$

Summing it in $n$ we have

$$y_{N+n} = y_{N-1} + n \Delta y_{N-1} + \sum_{i=0}^{n-1} \sum_{j=0}^{N+n-1} P_j y_{j-m}$$

(2.2)

The conclusions of Lemma 2.1 follow from (2.2).

From (2.2), the following is also true.

**LEMMA 2.2.** If

$$\lim_{n \to \infty} \sum_{i=0}^{n} P_i = \infty,$$

then

$$y_i \geq 0, \ i \in E_N, \ y_N \geq y_{N-1}$$

imply that $\{y_n\} \in S^+$, and if

$$y_i \leq 0, \ i \in E_N, \ y_N \leq y_{N-1}$$

imply that $\{y_n\} \in S^-$. 

**LEMMA 2.3.** Assume that the solution $y_n$ and $z_n$ have same initial values on $E_N$ with

$$\Delta y_{N-1} > \Delta z_{N-1}.$$ Then $y_n > z_n$, $\Delta y_n > \Delta z_n$, $n \geq N$ and

$$\lim_{n \to \infty} (y_n - z_n) = \infty.$$
PROOF. Set \(x_n = y_n - z_n\), then \(x_0 = 0\) on \(E_N\) and \(\Delta x_{N-1} > 0\). By Lemma 2.1, \(\{x_n\} \in S^{+\infty}\)
From (2.1) \(\Delta x_n > 0\) for \(n \geq N\).

**Lemma 2.4.** For every given initial value on \(E_N\), equation (1.1) has no more than one bounded solution.

**Proof.** Suppose the contrary, let \(\{y_n\}, \{z_n\}\) be two bounded solutions of (1.1) with \(y_i = z_i\) on \(E_N\) and \(y_N > z_N\). This implies that \(|y_n - z_n|\) is bounded. On the other hand, by Lemma 2.3, (2.4) should be true. This contradiction proves Lemma 2.4.

For given \(y_i = A_i\) on \(E_N\), then the solution of (1.1) depends on the parameter \(y_N = \xi \in R\).

**Define the sets of \(\xi\) as follows:**

\[
K^{+\infty} = \{\xi \in R, \{y_n\} \in S^{+\infty}\}
\]

\[
K^{-\infty} = \{\xi \in R, \{y_n\} \in S^{-\infty}\}
\]

\[
K^0 = \{\xi \in R, \{y_n\} \in S^0\}
\]

\[
K^- = \{\xi \in R, \{y_n\} \in S^-\}
\]

**Theorem 2.1.** For given \(y_i\) on \(E_N\), the sets \(K^{+\infty}\) and \(K^{-\infty}\) are nonempty.

**Proof.** If \(y_i = 0\) on \(E_N\), the conclusion follows from Lemma 2.1. Otherwise, from (2.1) and (2.2) we can find a number \(y_N = \xi \) so large that \(y_i > 0, i = N, N+1, \ldots, N+m\) and \(\Delta y_{N+m} > 0\). Translating the initial point to \(n+m\) and using Lemma 2.1 we conclude that the solution with this \(y_N\) belongs to \(S^{+\infty}\). Therefore \(\xi \in K^{+\infty}\). It is similar to prove that \(K^{-\infty}\) is nonempty.

**Theorem 2.2.** The sets \(K^{-\infty}, K^{+\infty}\) are open sets which are given by nonintersecting half lines \((-\infty, a)\) and \((\beta, +\infty)(a \leq \beta)\). The set \(F = R - (K^{+\infty} \cup K^{-\infty})\) is nonempty and consists of the interval \([a, \beta]\), if \(a < \beta\), or the point \(a\), if \(a = \beta\).

**Proof.** Let \(\{y_n\} \in S^{+\infty}\). Then there exists \(N'\) such that \(y_i > 0\) and \(\Delta y_i > 0\) on \(E_{N'}\). By continuous dependence of solutions and their differences on the initial conditions, all solutions with \(y_i\) on \(E_N\) and \(\overline{y}_N\) differ slightly from \(y_N\) are positive and have positive differences on \(E_{N'}\). If the initial point is translated to the point \(i = N'\), then by Lemma 2.1 all those solutions belong to \(S^{+\infty}\), i.e., \(K^{+\infty}\) is open. Similarly, one can prove that \(K^{-\infty}\) is open. Using Lemma 2.3, the conclusions of theorem follow.

**Theorem 2.3.** If \(a < \beta\), then each \(y_N \in F\) the corresponding solution is unbounded and oscillatory.

**Proof.** It is sufficient to show that every solution with \(y_N \in F\) is unbounded. Suppose the contrary, \(\{y_n\}\) is a bounded solution with \(y_N \in F\). Let \(z_N \neq y_N\). By Lemma 2.4, \(\{z_n\}\) is unbounded and oscillatory. On the other hand, Lemma 2.3 shows that \(|y_n - z_n| \to \infty\) as \(n \to \infty\) and hence \(\lim_{n \to \infty} |z_n| = \infty\) which contradicts the oscillation of \(\{z_n\}\).

**Theorem 2.4.** If \(\sum_{i=N}^{\infty} iP_i = \infty\), then every bounded solution of (1.1) either belongs to \(S^0\) or \(S^-\).

**Proof.** Let \(\{y_n\}\) be a bounded positive solution of (1.1). Then

\[
\Delta y_n < 0 \text{ eventually and } \lim_{n \to \infty} \Delta y_n = 0.
\]

From (2.1)

\[
\Delta y_{N-1} = \sum_{i=N-1}^{\infty} P_i y_{i-m}
\]

and from (2.2)
\[
y_{N+n} = y_{N-1} - n \sum_{i=N}^{\infty} p_{iy_{i-m}} + \sum_{i=0}^{n-1} j \sum_{i=N}^{n} p_{jy_{j-m}}
\]

\[
y_{N-1} - n \sum_{i=N}^{\infty} p_{iy_{i-m}} + \sum_{i=0}^{n-2} (n + N - 1 - i)p_{iy_{i-m}}
\]

\[
y_{N-1} - n \sum_{i=N}^{\infty} p_{iy_{i-m}} + \sum_{i=0}^{n-2} (N + n - 2) p_{iy_{i-m}}
\]

\[
y_{N-1} - n \sum_{i=N}^{\infty} p_{iy_{i-m}} + (N + n - 2) \sum_{i=0}^{n-2} p_{iy_{i-m}} - \sum_{i=N}^{n-2} (N + n - 2) p_{iy_{i-m}}
\]

\[
y_{N-1} + (N-1)(\Delta y_{N+n-2} - \Delta y_{N-1}) - \sum_{i=N}^{n-2} i p_{iy_{i-m}} + n \Delta y_{N+n-1}
\]

\[
\leq y_{N-1} - (N-1) \Delta y_{N-1} - \sum_{i=N}^{n-2} i p_{iy_{i-m}}.
\]  

(2.5)

If \(y_n \rightarrow l > 0\), then (2.5) lead to that \(\lim_{n \to \infty} y_n = -\infty\). This contradiction shows that \(\lim_{n \to \infty} y_n = 0\). The proof is complete.

**COROLLARY 2.1.** If \(\sum_{i=N}^{\infty} p_i = \infty\), then

\[
R = K^+ \cup K^- \cup K^0 \cup K^\sim
\]

and \(K^+, K^-, K^0\) and \(K^\sim\) are nonempty.

**THEOREM 2.5.** Assume that

\[
\lim_{n \to \infty} \sup_{n=m+1} (i - (n-m))p_i > 1
\]

(2.7)

Then every bounded solution of (1.1) is oscillatory.

**PROOF.** Let \(\{y_i\}\) be a bounded positive solution of (1.1). Then \(\Delta y_n < 0\) eventually. Summing (1.1) from \(N\) to \(n\), we have

\[
\Delta y_{n+1} - \Delta y_N = \sum_{i=N}^{n} p_i y_{i-m}
\]

Summing it from \(n-m+1\) to \(n\) in \(N\), we obtain

\[
m \Delta y_{n+1} - y_{n+1} + y_{n-m+1} = \sum_{j=n-m+1}^{n} \sum_{i=j}^{n} p_i y_{i-m}
\]

Hence

\[
0 \leq y_{n+1} - y_{n-m+1} + \sum_{i=n-m+1}^{n} (i - (n-m)) p_i y_{i-m}
\]

\[
\leq y_{n+1} - y_{n-m+1} (1 - \sum_{i=n-m+1}^{n} (i - (n-m)) p_i)
\]

which contradicts to (2.7). The proof is complete.

**COROLLARY 2.2** Assume that the assumptions of Corollary 2.1 and Theorem 2.5 hold.
Then $\kappa^-$ is nonempty.

In fact, by Corollary 2.1, $\kappa^o \cup \kappa^-$ is nonempty and by Theorem 2.5, $\kappa^o$ is empty. Therefore $\kappa^-$ is nonempty.

It is easy to see that if $p_i \equiv p > 0$ in (1.1), then all assumptions of Corollary 2.2 hold, therefore for any given $A_n$ on $E_N$, equation (1.1) with (1.3) has at least one oscillatory solution, i.e., $\kappa^-$ is nonempty.

**EXAMPLE 2.1.** Consider

$$\Delta^2 u_n = p_n u_{n-4}$$

with $u_i = (-1)^i, i = 1, \ldots, 5, p_n \equiv 1$. Then through computation if $u_0 > -0.21675$, the solution $(u_n) \in S^+$, if $u_0 < -0.21676$, the solution $(u_n) \in S^-$, in this case we see $\alpha = \beta$.

**OPEN PROBLEM.** What condition could guarantee that $\alpha < \beta$?

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