ON SOME REGULAR AND SINGULAR PROBLEMS OF BIRKHOFF INTERPOLATION

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ABSTRACT. Here we investigate the pure \((0,1,\ldots,r-2,r)\)-interpolation problem on the zeros of \((1-z^2) P_n^{(\alpha,\beta)}(z) = (1-z^2) P_n^{(\alpha)}(z)\), \(\alpha > -1\), where \(P_n^{(\alpha,\beta)}(x)\) is the Jacobi polynomial of degree \(n\) with \(\beta = \alpha\).

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1. INTRODUCTION.

Let \(k\) and \(l\) be natural numbers and let \(E = E_k^l = (e_{ij})\) \((i = 1,2,\ldots,k; j = 0,1,\ldots,l-1)\) be a matrix with \(k\) rows and \(l\) \((l \geq k)\) columns having \(e_{ij}\) 0 or 1, which are such that \(\sum e_{ij} = 1\) and no row is entirely composed of zeros. Let \(x_1 < x_2 < \cdots < x_k\) \((1.1)\) be increasing reals and \(e_i^k = \{(i,j) : e_{ij} = 1\}\). The reals \(x_i\) and the incidence matrix \(E\) describe the interpolation problem

\[ p^{(j)}(x_i) = y^{(j)}, \text{ for } (i,j) \in e_i^k \] \((1.2)\)

where \(y^{(j)}\) are prescribed and the problem is to find the polynomial \(P(x)\) of degree \(\leq l-1\), which satisfies the condition \((1.2)\). If \(y^{(j)} = 0\) for \((i,j) \in e_i^k\), then the problem \((1.2)\) is the homogeneous interpolation problem. Let \(X = \{x_i\}\) be the interpolation nodes. We say that \((E,X)\) is regular if \((1.2)\) has a unique solution for all choices of reals \(y^{(j)}\), and singular otherwise. If \(p^{(j)}(x_i) = 0\) for \((i,j) \in e_i^k\), then \(P(x)\) is said to be annihilated by \((E,X)\).

Turán and his associate [4] considered \(E = E_{2n}^n\) with \(x_1, x_2, \ldots, x_n\) as the zeros of \(P_n(x) = (1-x^2)P_n-1(x)\), where \(P_n(x)\) is the Legendre polynomial of degree \(n\) with normalization \(P_n(1) = 1\). Turán proved that \((E,X)\) is regular if \(n\) is even and singular if \(n\) is odd. Later, Varma ([5], [6]); Anderson and Prasad [1]; and Prasad and Anderson [3] considered different incidence matrices. Recently, Bajpai and Saxena [2] proved the following:

**THEOREM A.** If \(E\) is the matrix of order \((n + 2) \times (m + 1)(n + 2)\), \(m \geq 2\), with rows \((1,1,\ldots,1,0,0,\ldots,0)\) and \(X\) is the set of zeros of \((1-x^2)P_n(x)\), \(P_n(x)\) being the Legendre polynomial of degree \(n\), then:

(i) if \(m\) is even, \((E,X)\) is singular, and
(ii) if \( m \) is odd, \((E, X)\) is regular if \( n \) is even and singular if \( n \) is odd.

Let \( X \) be the set of the zeros \( \{z_k\}_{k=1}^{n+1} \) of \((1 - z^2) P_n^{(\alpha)}(x) = (1 - z^2) P_n^{(\alpha, \alpha)}(z)\), \( \alpha > -1 \), where \( P_n^{(\alpha, \alpha)}(z) \) is the Jacobi polynomial of degree \( n \) with \( \beta = \alpha \), such that

\[-1 = x_{n+1} < x_n < \cdots < x_1 < x_0 = 1.\]

Our aim here is to prove the following:

**THEOREM 1.** Let \( X \) be the set of the zeros of \((1 - z^2) P_n^{(\alpha)}(z), \alpha > -1, \) and \( E \) be the incidence matrix given by

\[
E = E_{n+2}^{(n+2) \times (m+1)(n+2)} = \\
\begin{pmatrix}
(1)_{m} & 0 & 1 & 0 & \cdots & 0 \\
(1)_{m} & 0 & 1 & 0 & \cdots & 0 \\
& \cdots & \cdots & \cdots & \cdots & \cdots \\
(1)_{m} & 0 & 1 & 0 & \cdots & 0
\end{pmatrix}
\]

where \((1)_{m}\) means \( m \) entries of \( 1 \) in that row. Let \( m \) be an odd positive integer \( \geq 3 \), and \(-1 < \alpha < 1\), then:

(i) if \( n \) is odd then \((E, X)\) is singular.

(ii) if \( n \) is even, \( \alpha \neq \frac{m - 2}{m + 2} \), and \( \alpha \) is such that \( m - 1 - \alpha(m + 2) \) is an odd positive integer then \((E, X)\) is singular and for all other values of \( m - 1 - \alpha(m + 2) \), \((E, X)\) is regular.

(iii) if \( n \) is even and \( \alpha = \frac{m - 2}{m + 2} \), then \((E, X)\) is singular.

**THEOREM 2.** Let \( X \) be the set of the zeros of \((1 - z^2) P_n^{(\alpha)}(z), \alpha > -1, \) and \( E \) be the incidence matrix given by (1.1). Let \( m \) be an even positive integer \( \geq 2 \), and \(-1 < \alpha < 1\), then:

(i) If \( n \) is odd then \((E, X)\) is singular.

(ii) If \( n \) is even and \( \alpha \neq \frac{m - 2}{m + 2}, 0 \leq \alpha < 1\), then \((E, X)\) is singular.

(iii) If \( n \) is even and \( \alpha = \frac{m - 2}{m + 2} \), then \((E, X)\) is singular if \( m - 1 - \alpha(m + 2) \) is an odd positive integer and regular otherwise.

2. SOME LEMMAS.

Here we state and prove a few lemmas.

**LEMMA 1.** If \( w_n(x) = P_n^{(\alpha)}(x), \alpha > -1, \lambda_r(x) = [(1 - x^2)w_n^2(x)]^r, r = 1, 2, \cdots \) and \( \{z_k\}_k^n \) are the zeros of \( w_n(x) \) then:

\[
[w_n^{2r}(x)]_{x=z_k} = (2r)! \left[w_n'(x_k)\right]^{2r} \tag{2.1}
\]

\[
[w_n^{2r+1}(x)]_{x=z_k} = 2r (2r + 1)! \left(1 - z_k^2\right)^{-1} \left[w_n'(x_k)\right]^{2r} \tag{2.2}
\]

\[
\lambda_r^{(i)}(x_k) = \begin{cases} 
0, & i = 0, 1, 2r - 1, \\
(1 - z_k^2)^r (2r)! \left[w_n'(x_k)\right]^{2r}, & i = 2r,
\end{cases} \tag{2.3}
\]

\[
\lambda_r^{(2r+1)}(x_k) = 2r (2r + 1)! \left[\alpha x_k (1 - z_k^2)^{r-1} \left[w_n'(x_k)\right]^{2r}\right] \tag{2.4}
\]

The proof is obvious.

**LEMMA 2.** Let \( \delta_{2r}(x) = (1 - x^2)^{2r} = (x^2 - 1)^{2r}, r = 1, 2, \cdots \).

Then:

\[
\delta_r^{(i)}(\pm 1) = \begin{cases} 
0, & i = 0, 1, \cdots, 2r - 1, \\
(2r)! 2^{2r}, & i = 2r
\end{cases} \tag{2.5}
\]
\[
\delta_{2r}^{(2r+1)}(1) = 2^{2r} (2r+1)! \delta_{2r}^{(2r+1)}(-1)
\]
(2.6)
\[
\delta_{2r}^{(2r+1)}(1) = r (2r+1) \delta_{2r}^{(2r)}(1)
\]
(2.7)
and
\[
\delta_{2r}^{(2r+1)}(-1) = -r (2r+1) \delta_{2r}^{(2r)}(-1).
\]
(2.8)

The proof is obvious.

**Lemma 3.** Let \( F_n(x) = [(1-x^2)u_n(x)]^m q_{n+1}(x) \) be a polynomial of degree \( \leq (n+2)(m+1) - 1 \), where \( q_{n+1}(x) \) is a polynomial of degree \( \leq n+1 \), and let
\[
F_n^{(m+1)}(x_k) = 0, k = 0, 1, 2, \ldots, n+1.
\]

Then, \( q_{n+1}(x) \) satisfies the following conditions:
\[
(1-x^2) q_n''(x_k) + m(\alpha-1)x_k q_n'(x_k) = 0, \quad k = 1, 2, \ldots, n; \quad \alpha > -1,
\]
(2.9)
\[
2q_n'' + n(n+2\alpha+1)/(1+\alpha) q_n' + n(n+2\alpha+1)/(1+\alpha) q_n = 0.
\]
(2.10)

**Proof.** Let \( m = 2r \). Then
\[
F_n(x) = \lambda_n(x) \left[ (1-x^2)^m q_{n+1}(x) \right].
\]

On using Leibnitz’s formula and Lemma 1 one can easily see that for \( k = 1, 2, \ldots, n, \)
\[
F_n^{(2r+1)}(x_k) = (2r+1) x_r^{(2r)}(x_k) (1-x^2)^{r-1} \left[ (1-x^2) q_n''(x_k) + 2r x_k (\alpha-1) q_n'(x_k) \right].
\]
(2.12)

To evaluate \( F_n^{(2r+1)}(\pm 1) \) we proceed as follows:
\[
F_n(x) = \delta_{2r}(x) \left[ [w_n(x)]^{2r} q_{n+1}(x) \right].
\]

Now, making use of Leibnitz formula and Lemma 2, we get
\[
F_n^{(2r+1)}(1) = \delta_{2r}^{(2r+1)}(1) [w_n(1)]^{2r} q_{n+1}(1) +
\]
\[
+ \left( \frac{2r+1}{2r} \right) \delta_{2r}^{(2r)}(1) \left[ 2r [w_n(1)]^{2r-1} w_n'(1) q_{n+1}(1) + [w_n(1)]^{2r} q_n'(1) \right].
\]
(2.13)

We know that
\[
(1-x^2) w_n''(x) - 2(\alpha+1)x w_n'(x) + n(n+2\alpha+1) w_n(x) = 0
\]
(2.14)

hence
\[
w_n'(1) = \frac{n(n+2\alpha+1)}{2(\alpha+1)} w_n(1), \quad w_n(1) = \left( \frac{n+\alpha}{n} \right)
\]
(2.15)

So, from (2.13) and (2.15) it follows that
\[
F_n^{(2r+1)}(1) = 2^{2r-1} (2r+1) [w_n(1)]^{2r} \left[ 2q_n''(1) + 2r \left[ 1 - \frac{n(n+2\alpha+1)}{1+\alpha} \right] q_n'(1) \right].
\]
(2.16)

We also know that
\[
w_n(-1) = (-1)^n w_n(1),
\]
(2.17)
Further, using Leibnitz formula, Lemma 2, (2.17) and (2.18) one can easily verify that

\[ F_n^{(2r + 1)}(-1) = 2^{2r-1}(2r+1)! w_n(-1) \left\{ 2q_n^{(1)}(-1) - 2r \left[ \frac{n(n+2\alpha+1)}{1+\alpha} \right] q_{n+1}^{(1)}(-1) \right\} \]

Next, let \( m = 2r + 1 \). We now write

\[ F_n(x) = \lambda_n(x)(1 - x^2)^r + 1 w_n(x) q_{n+1}(x) \]

Again, on using Leibnitz formula, Lemma and (2.14), it follows that for \( r = 1, 2, \ldots, n \),

\[ F_n^{(2r+2)}(x) = (2r+1)(2r+2)(1-x^2) w_n'(x) \lambda_n^{(2r)}(x) \left[ (1-x^2) q_n^{(1)}(x) \right] + (2r+1)(\alpha-1)x_k q_{n+1}(x) \]

Further, to compute \( F_n^{(2r+2)}(\pm 1) \), we write

\[ F_n(x) = \delta_{2r}(x)(1-x^2) w_n^{2r+1}(x) q_{n+1}(x) \]

and use Leibnitz formula to get

\[ F_n^{(2r+1)}(x) = \sum_{i=0}^{2r+2} \binom{2r+2}{i} \delta_{2r}^{(i)}(x) \left[ (1-x^2) w_n^{2r+1}(x) q_{n+1}(x) \right] (2r+1-i). \]  

On simplification using Lemma 2, (2.21) yields

\[ F_n^{(2r+2)}(1) = -(2r+2)! 2^r w_n^{2r+1}(1) \left[ 2q_n^{(1)}(1) + (2r+1) \left\{ \frac{n(n+2\alpha+1)}{1+\alpha} + 1 \right\} q_{n+1}(1) \right] \]

\[ F_n^{(2r+2)}(-1) = (-1)^n(2r+2)! 2^r w_n^{2r+1}(-1) \left[ 2q_n^{(1)}(-1) - (2r+1) \left\{ \frac{n(n+2\alpha+1)}{1+\alpha} + 1 \right\} q_{n+1}(-1) \right] \]

Hence the conditions

\[ F^{(m+1)}(x_k) = 0, \quad k = 0, 1, 2, \ldots, n+1 \]

along with (2.12), (2.16), (2.19), (2.20), (2.22) and (2.23) imply (2.9), (2.10) and (2.11) for \( m \) even or odd. This completes the proof of Lemma 3.

**Lemma 4.** Let \( q_{n+1}(x) \) be a polynomial of degree \( \leq n+1 \) which satisfies the following \( n+2 \) conditions:

\[ (1-x^2) q_n^{(1)}(x) + m(\alpha-1) x_k q_{n+1}(x) = 0, \quad k = 1, 2, \ldots, n; \alpha > -1, \]

\[ 2q_n^{(1)}(1) + m \left[ \frac{n(n+2\alpha+1)}{1+\alpha} + 1 \right] q_{n+1}(1) = 0, \]

\[ 2q_n^{(1)}(-1) - m \left[ \frac{n(n+2\alpha+1)}{1+\alpha} + 1 \right] q_{n+1}(-1) = 0. \]

Then \( q_{n+1}(x) \) satisfies the following equation:

\[ (1-x^2) q_{n+1}(x) + m(\alpha-1) x_k q_{n+1}(x) = c[x^2 - \Delta(\alpha)] w_n(x), \]

where \( c \) is an arbitrary constant and

\[ \Delta(\alpha) = \frac{n(n+2\alpha+1)+\alpha+1}{1+\alpha} \left[ \frac{n(n+2\alpha+1)+\alpha+1}{1+\alpha} \right] \left[ \frac{n(n+2\alpha+1)+\alpha+1}{1+\alpha} \right] - \frac{n(n+2\alpha+1)+\alpha+1}{1+\alpha} \left[ \frac{n(n+2\alpha+1)+\alpha+1}{1+\alpha} \right]. \]
PROOF. Due to (2.24), it follows that
\[(1 - x^2)\gamma_n + 1(x) + m(\alpha - 1)\gamma_n + 1(x) = (cx^2 + dx + e)w_n(x),\]
(2.29)
where \(c, d\) and \(e\) are constants. From (2.29), (2.15) and (2.17) we see that
\[m(\alpha - 1)\gamma_n + 1(1) = (c + d + e)\left(\frac{n + \alpha}{n}\right),\]
(2.30)
\[-m(\alpha - 1)\gamma_n + 1(- 1) = (c - d + e)(- 1)^n\left(\frac{n + \alpha}{n}\right).\]
(2.31)
Also, on differentiating (2.29) we have
\[(1 - x^2)\gamma_n + 1(x) + [m(\alpha - 2)\gamma_n + 1(x) + m(\alpha - 1)\gamma_n + 1(x) =
(cx^2 + dx + e)w_n(x) + (2cx + d)w_n(x).
(2.32)
Hence, from (2.32) we conclude that
\[[m(\alpha - 1) - 2]\gamma_n + 1(1) + m(\alpha - 1)\gamma_n + 1(1) = (c + d + e)w_n(1) + (2c + d)w_n(1),\]
(2.33)
\[-[m(\alpha - 1) - 2]\gamma_n + 1(- 1) + m(\alpha - 1)\gamma_n + 1(- 1) = (c - d + e)w_n(-1) + (- 2c + d)w_n(-1).\]
(2.34)
Further, from (2.30), (2.31), (2.15), (2.17), (2.18), (2.33) and (2.34) it follows that
\[[m(\alpha - 1) - 2]\gamma_n + 1(1) = (c - e)\left(\frac{n + \alpha}{n}\right) + (c + d + e)\frac{n(n + 2\alpha + 1)}{2(\alpha + 1)},\]
(2.35)
\[[m(\alpha - 1) - 2]\gamma_n + 1(- 1) = (- 1)^n\left[(c - e)\left(\frac{n + \alpha}{n}\right) + (c - d + e)\frac{n(n + 2\alpha + 1)}{2(\alpha + 1)}\right].\]
(2.36)
Consequently, on substituting the values of \(\gamma_n + 1(1), \gamma_n + 1(1), \gamma_n + 1(- 1)\) and \(\gamma_n + 1(- 1)\) from the above equations into (2.25) and (2.26) and simplifying we get
\[\left[\left(\frac{n + \alpha + 1}{n}\right) - 1\right]c - d - \left(\frac{n + \alpha + 1}{n}\right)c + (c + d + e) + \left(\frac{n + \alpha + 1}{n}\right)c = 0,\]
(2.37)
\[\left[\left(\frac{n + \alpha + 1}{n}\right) - 1\right]c + d - \left(\frac{n + \alpha + 1}{n}\right)c + (c - d + e) + \left(\frac{n + \alpha + 1}{n}\right)c = 0.\]
(2.38)
Now, from (2.37) and (2.38) we see that \(d = 0\) and
\[\left(\frac{n + \alpha + 1}{n}\right) - 1\]c - \left(\frac{n + \alpha + 1}{n}\right)c + (c + e)\left[\frac{n(n + 2\alpha + 1)}{1 + \alpha}\right]\left[\frac{m(1 - \alpha) + 2}{1 - \alpha}\right] + 1 = 0
which, on simplification, yields
\[e = \frac{n(n + 2\alpha + 1) + \alpha + 1}{n(n + 2\alpha + 1) + \alpha + 1}\left[\frac{m(1 - \alpha) + 2}{1 - \alpha}\right] + 1 + \left[\frac{2(n + \alpha + 1)}{n + \alpha + 1}\right]c\]
(2.39)
or, using (2.28) we have

\[ \epsilon = -\Delta(\alpha)c. \]  

(2.40)

This completes the proof of Lemma 4.

3. PROOF OF THEOREM 1 AND THEOREM 2.

Let \( E \) be the incidence matrix given by (1.3) and let \( X \) be the set of zeros of \((1 - z^2)P_{n}^{(\alpha)}(z) = (1 - z^2)w_n(z)\). Let \( F_n(x) \) be a polynomial of degree \( \leq (n + 2)(m + 1) - 1 \) annihilated by \((E, X)\). We have to ascertain if \( F_n(x) \) is identically zero. Since

\[
F_n(x_k) = F'_n(x_k) = \cdots = F^{(m - 1)}_n(x_k) = 0, \quad k = 0, 1, \ldots, n + 1,
\]

\[
F_n(x) = [(1 - z^2)w_n(z)]^{m + 1}q_{n + 1}(x),
\]

where \( q_{n + 1}(x) \) is a polynomial of degree \( \leq n + 1 \). Further, since we have required that

\[
F^{(m + 1)}_n(x_k) = 0, \quad k = 0, 1, \ldots, n + 1,
\]
on account of Lemma 4, \( q_{n + 1}(x) \) satisfies the following equation:

\[
(1 - z^2)w_{n + 1}(x) + m(a - 1)zw_{n + 1} = c[z^2 - \Delta(\alpha)]w_n(x),
\]

(3.1)

where \( c \) is a numerical constant. Let

\[ q_{n + 1}(x) = \sum_{k=0}^{n+1} a_kw_k(x). \]  

(3.2)

Further, it is well-known that

\[
(1 - z^2)w'_n(x) = -nzw_n(x) + (n + \alpha)w_{n - 1}(x).
\]

(3.3)

Now, from (3.1), (3.2) and (3.3), on simple computations, it follows that

\[
\sum_{k=1}^{n+2} a_{k - 1} \frac{m(\alpha - 1) - k + 1}{(k + 2)(2k + 2\alpha)} w_k(x) + \sum_{k=0}^{n} a_{k + 1}(k + \alpha + 1) \left[ 1 + \frac{m(\alpha - 1) - k - 1}{2k + 2\alpha + 3} \right] w_k(x) = c[z^2 - \Delta(\alpha)]w_n(x).
\]

(3.4)

Also, we know that

\[ xw_n(x) = \frac{(n + 1)(n + 2\alpha + 1)}{(2n + 2\alpha + 1)(n + \alpha + 1)} w_{n + 1}(x) + \frac{(n + \alpha)}{2n + 2\alpha + 1} w_{n - 1}(x). \]  

(3.5)

Repeated application of (3.5) in (3.4), on simplification, yields

\[
\sum_{k=0}^{n} a_{k + 1}(k + \alpha + 1) \frac{\frac{k + \alpha(m + 2) - 2 - m}{2k + 2\alpha + 3}}{w_k(x)} + \sum_{k=1}^{n+2} a_{k - 1} \frac{\frac{m(\alpha - 1) - k + 1}{(k + \alpha)(2k + 2\alpha - 1)}}{w_k(x)}
\]

\[ = A w_{n - 2}(x) + B w_n(x) + C w_{n + 2}(x), \]

(3.6)

where

\[ A = \frac{(n + \alpha)(n + \alpha - 1)}{(2n + 2\alpha + 1)(2n + 2\alpha - 1)} c, \]

(3.7)

\[ B = \left[ \frac{(n + 1)(n + 2\alpha + 1)}{(2n + 2\alpha + 1)(2n + 2\alpha + 3)} + \frac{n(n + 2\alpha)}{(2n + 2\alpha + 1)(2n + 2\alpha - 1)} \right] + \Delta(\alpha) c. \]  

(3.8)
Consequently, we obtain

\[
(1 + \alpha) \left[ \alpha(m + 2) + 2 - m \right] a_1 = 0,
\]

\[
\frac{(k + 1 + \alpha) \left[ k + \alpha(m + 2) + 2 - m \right]}{(2k + 2\alpha + 3)} a_{k+1} + \frac{k(k + 2\alpha) \left[ m(\alpha - 1) - k + 1 \right]}{(k + \alpha)(2k + 2\alpha - 1)} a_{k-1} = 0,
\]

\[k = 1, 2, \ldots, n-4, n-3,\]

\[
\frac{(n + \alpha - 1) \left[ n + \alpha(m + 2) - m \right]}{(2k + 2\alpha - 1)} a_{n-1} + \frac{m(\alpha - 1) - n + 3}{(n + \alpha - 2)(n + 2\alpha - 2)} a_{n-3} = A,
\]

\[
\frac{(n + \alpha) \left[ n + \alpha(m + 2) + 1 - m \right]}{(2n + 2\alpha + 1)} a_n + \frac{m(\alpha - 1) - n + 2}{(n + \alpha - 1)(2n + 2\alpha - 1)} a_{n-2} = 0,
\]

\[
\frac{(n + \alpha + 1) \left[ n + \alpha(m + 2) + 2 - m \right]}{(2n + 2\alpha + 3)} a_{n+1} + \frac{m(\alpha - 1) - n + 1}{(n + \alpha)(2n + 2\alpha - 1)} a_{n-1} = B,
\]

\[
\frac{(n + 2\alpha + 1) \left[ n(\alpha - 1) - n \right]}{(n + \alpha + 1)(2n + 2\alpha + 1)} a_n = 0,
\]

and

\[
\frac{(n + 2) \left[ n + 2\alpha + 2 \right]}{(n + \alpha + 2)(2n + 2\alpha + 3)} a_{n+1} = C.
\]

Let \( m \) be an odd positive integer \( \geq 3 \):

(i) If \( n \) is odd, \( -1 < \alpha < 1 \), \( \alpha \) is such that \( m - 1 - \alpha(m + 2) \) is an even positive integer then

\[a_n = a_{n-2} = \cdots = a_3 = a_1 = 0\]

and

\[a_0 = a_2 = \cdots = a_{m-3} = a_{m-1} = 0\]

but \( a_{m-1} - \alpha(m + 2)a_m + 1 - \alpha(m + 2) \cdot \cdots \cdot a_{n+1} \) are not necessarily zero. Hence, \( q_{n+1}(x) \) is not identically zero. If \( n \) is odd, \( -1 < \alpha < 1 \), \( \alpha \) is such that \( m - 1 - \alpha(m + 2) \) is an even negative integer then

\[a_n = a_{n-2} = \cdots = a_3 = a_1 = 0\]

and \( a_0, a_2, \cdots, a_{n-3} \) are not necessarily zero. Hence, \( q_{n+1}(x) \) is not identically zero.

If \( n \) is odd, \( -1 < \alpha < 1 \), \( \alpha \) is such that \( m - 1 - \alpha(m + 2) \) is an odd integer or a fraction then

\[a_n = a_{n-2} = \cdots = a_3 = a_1 = 0\]

but \( a_0, a_2, \cdots, a_{n-3} \) and \( a_{n+1} \) are not all zero. Hence, \( q_{n+1}(x) \) is not identically zero.

So, it follows that \( (E, X) \) is singular if \( n \) is odd.

(ii) If \( n \) is even, \( -1 < \alpha < 1, \alpha \neq \frac{m-2}{m+2} \) and \( \alpha \) is such that \( m - 1 - \alpha(m + 2) \) is an odd positive integer then

\[a_n = a_{n-2} = \cdots = a_2 = a_0 = 0\]

and
\[ a_1 = a_3 = \cdots = a_{m-3} - \alpha(m+2) = 0, \]
but \( a_{m-1} - \alpha(m+2), a_{m+1} - \alpha(m+2), \ldots, a_{n+1} \) are not necessarily zero. Hence, \( q_{n+1}(x) \) is not identically zero. If \( n \) is even, \( -1 < \alpha < 1, \alpha \neq \frac{m-2}{m+2} \), \( \alpha \) is such that \( m-1 - \alpha(m+2) \) is an odd negative integer then
\[ a_n = a_{n-2} = \cdots = a_2 = a_0 = 0. \]
Noting that \( a_1 = 0 \) we conclude that
\[ a_1 = a_3 = \cdots = a_{n-3} = 0. \]
Recalling the equations for \( a_n \) and \( a_{n+1} \) and substituting the values of \( A, B, \) and \( C \) it can be easily verified that \( c = 0 \). So, \( a_{n-1} \) and \( a_{n+1} \) are also zero. Hence, \( q_{n+1}(x) \) is identically zero. If \( n \) is even, \( -1 < \alpha < 1, \alpha \neq \frac{m-2}{m+2} \) and \( \alpha \) is such that \( m-1 - \alpha(m+2) \) is an even positive integer then
\[ a_n = a_{n-2} = \cdots = a_2 = a_0 = 0 \]
and
\[ a_1 = a_3 = \cdots = a_{n-3} = a_{n-1} = a_{n+1} = 0. \]
Hence, \( q_{n+1}(x) \) is identically zero.

If \( n \) is even, \( -1 < \alpha < 1, \alpha \neq \frac{m-2}{m+2} \) and \( \alpha \) is such that \( m-1 - \alpha(m+2) \) is an even negative integer then
\[ a_n = a_{n-2} = \cdots = a_2 = a_0 = 0 \]
and
\[ a_1 = a_3 = \cdots = a_{n-3} = a_{n-1} = a_{n+1} = 0. \]
Hence, \( q_{n+1}(x) \) is identically zero.

If \( n \) is even, \( -1 < \alpha < 1, \alpha \neq \frac{m-2}{m+2} \) and \( \alpha \) is such that \( m-1 - \alpha(m+2) \) is a fraction then
\[ a_n = a_{n-2} = \cdots = a_2 = a_0 = 0 \]
and
\[ a_1 = a_3 = \cdots = a_{n-3} = a_{n-1} = a_{n+1} = 0. \]
Hence, \( q_{n+1}(x) \) is identically zero. Consequently, if \( n \) is even, \( \alpha = \frac{m-2}{m+2} \) and \( \alpha \) is such that \( m-1 - \alpha(m+2) \) is an odd positive integer then \((E, X)\) is singular and for all other values of \( m-1 - \alpha(m+2), (E, X) \) is regular.

(iii) If \( n \) is even, \( \alpha = \frac{m-2}{m+2} \), and \( \alpha \) is such that \( m-1 - \alpha(m+2) \) is a negative integer then
\[ a_n = a_{n-2} = \cdots = a_2 = a_0 = 0 \]
and \( a_1, a_3, \ldots, a_{n-3} \) are not necessarily zero. Hence, \( q_{n+1}(x) \) is not identically zero.

If \( n \) is even, \( \alpha = \frac{m-2}{m+2} \) and \( \alpha \) is such that \( m-1 - \alpha(m+2) \) is an odd positive integer then
\[ a_n = a_{n-2} = \cdots = a_2 = a_0 = 0 \]
and
\[ a_1 = a_3 = \cdots = a_{m-3} - \alpha(m+2) = 0 \]
but \( a_{m-1} - \alpha(m+2), a_{m+1} - \alpha(m+2), \ldots, a_{n-1} \) are not necessarily zero. Hence, \( q_{n+1}(x) \) is not identically zero. If \( n \) is even, \( \alpha = \frac{m-2}{m+2} \) and \( \alpha \) is such that \( m-1 - \alpha(m+2) \) is an even positive integer then
\[
a_n = a_{n-2} = \cdots = a_2 = a_0 = 0
\]
and \( a_1, a_2, \ldots, a_{n-3} \) are not necessarily zero. Hence, \( q_{n+1}(x) \) is not identically zero. Consequently, in this case, \((E, X)\) is singular.

This completes the proof of Theorem 1.

Next, let \( m \) be an even positive integer \( \geq 2 \), and \(-1 < \alpha < 1\):

(i) If \( n \) is odd then
\[
a_n = a_{n-2} = \cdots = a_3 = a_1 = 0
\]
but not all \( a_0, a_2, \ldots, a_{n+1} \) are zero. Hence, \( q_{n+1}(x) \) is not identically zero. So, \((E, X)\) is singular.

(ii) If \( n \) is even and \( \alpha = \frac{m-2}{m+2} \), \( 0 \leq \alpha < 1 \), then
\[
a_n = a_{n-2} = \cdots = a_2 = a_0 = 0
\]
but \( a_1, a_3, \ldots, a_{n-3} \) are not necessarily zero. Hence \( q_{n+1}(x) \) is not identically zero. Consequently, \((E, X)\) is singular.

(iii) If \( n \) is even, \( \alpha \neq \frac{m-2}{m+2} \), and \( \alpha \) is such that \( m-1 - \alpha(m+2) \) is an odd positive integer then
\[
a_n = a_{n-2} = \cdots = a_2 = a_0 = 0
\]
and
\[
a_1 = a_3 = \cdots = a_{m-3} = a_{m-1} = a_{n+1} = 0,
\]
but \( a_{m-1} - \alpha(m+2), a_{m+1} - \alpha(m+2), \ldots, a_{n+1} \) are not necessarily zero. Hence \( q_{n+1}(x) \) is not identically zero. If \( n \) is even, \( \alpha \neq \frac{m-2}{m+2} \) and \( \alpha \) is such that \( m-1 - \alpha(m+2) \) is an odd negative integer then
\[
a_n = a_{n-2} = \cdots = a_2 = a_0 = 0
\]
and since \( k + \alpha(m+2) + 2 - m \) is never zero for even values of \( k \) hence
\[
a_1 = a_3 = \cdots = a_{n-3} = a_{n-1} = a_{n+1} = 0.
\]
So, \( q_{n+1}(x) \) is identically zero.

If \( n \) is even, \( \alpha \neq \frac{m-2}{m+2} \) and \( \alpha \) is such that \( m-1 - \alpha(m+2) \) is an even positive integer then
\[
a_n = a_{n-2} = \cdots = a_2 = a_0 = 0
\]
and also \( a_1 = 0 \) so that \( k + \alpha(m+2) + 2 - m \) is never zero for even values of \( k \) hence
\[
a_1 = a_3 = \cdots = a_{n-3} = a_{n-1} = a_{n+1} = 0.
\]
Consequently, \( q_{n+1}(x) \) is identically zero.

If \( n \) is even, \( \alpha \neq \frac{m-2}{m+2} \) and \( \alpha \) is such that \( m-1 - \alpha(m+2) \) is an even negative integer then
\[
a_n = a_{n-2} = \cdots = a_2 = a_0 = 0
\]
and also \( a_1 = 0 \) so that \( k + \alpha(m+2) + 2 - m \) is never zero hence
\[
a_1 = a_3 = a_2 = \cdots = a_{n-3} = a_{n-1} = a_{n+1} = 0.
\]
Consequently, \( q_{n+1}(x) \) is identically zero.

If \( n \) is even, \( \alpha \neq \frac{m-2}{m+2} \) and \( \alpha \) is such that \( m-1-\alpha(m+2) \) is a fraction then
\[
a_n = a_{n-2} = \ldots = a_0 = 0,
\]
\[
a_1 = 0, \text{ so } a_1 = a_3 = a_5 = \ldots = a_{n-3} = a_{n-1} = a_{n+1} = 0.
\]
Therefore, \( q_{n+1}(x) \) is identically zero. Consequently, \((E, X)\) is singular if \( m-1-\alpha(m+2) \) is an odd positive integer and regular otherwise. This completes the proof of Theorem 2.

In conclusion, it is worthwhile to mention that H. Windauer [7] has also considered the modified \((0,1,\ldots,r-2,r)\)-interpolation problem on the zeros of \((1-z^2)p^{(\alpha)}(z), \alpha > -1\), and \((0,1,\ldots,r-2,r)\)-interpolation problem on the zeros of \(p^{(\alpha)}(z), \alpha > -1\). As is evident, we have addressed the \((0,1,\ldots,r-2,r)\)-interpolation problem on the zeros of \((1-z^2)p^{(\alpha)}(z), \alpha > -1\).

REFERENCES