VECTOR-VALUED MEANS AND WEAKLY ALMOST PERIODIC FUNCTIONS

CHUANYI ZHANG

Department of Mathematics
University of British Columbia
Vancouver, B.C., Canada V6T 1Z2

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ABSTRACT. A formula is set up between a vector-valued mean and scalar-valued means that enables us translate many important results about scalar-valued means developed in [1] to vector-valued means. As applications of the theory of vector-valued means, we show that the definitions of a mean in [2] and [3] are equivalent and the space of vector-valued weakly almost periodic functions is admissible.

KEY WORDS AND PHRASES. Means, semigroup, weakly almost periodic functions

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Scalar-valued means have been much studied. However, little has been done on the vector-valued means. In this paper we develop the theory of vector-valued means.

In Lemma 1.4, we set up a formula between a vector-valued mean and scalar-valued means, by which we will be able to translate many important results about scalar-valued means developed in [1] to vector-valued means. We present these results in Sections 1, 2 and 3. As an application of the theory established in these sections, we investigate vector-valued weakly almost periodic functions in Section 4.

§1. Means on a Linear Subspace of $B(S, X)$

Throughout this paper, $S$ denotes a semigroup which need not have an identity, $X$ denotes a Banach space and $X^*$ is the dual space of $X$. $B(S, X)$ denotes all of the bounded functions from $S$ to $X$. When $X = C$, we simply write $B(S)$ for $B(S, X)$. $A$ denotes a linear subspace of $B(S, X)$ containing the constant functions. $L(A, X)$ denotes all of the bounded linear mappings from $A$ to $X$.

Let $f \in B(S, X)$. Then the right (respectively, left) translate $R_sf$ of $f$ by $s \in S$ is the map $R_sf(t) = f(ts)$ (respectively, $L_sf(t) = f(st)$) for all $t \in S$.

$A$ is said to be right (respectively, left) translation invariant if $R_SA = \{R_sf : s \in S, f \in A\} \subset A$ (respectively, $L_SA = \{L_sf : s \in S, f \in A\} \subset A$). $A$ is said to be translation invariant if it is both right and left translation invariant.

Definition 1.1 [2]. A linear mapping $\mu : A \rightarrow X$ is called a mean on $A$ provided $\mu(f) \in \overline{\text{co}} f(S)$, for all $f \in A$. Denote by $M(A)$ the set of all means on $A$.

If $A$ is right (respectively, left) translation invariant, $\mu$ is said to be right (respectively, left) invariant if $\mu(R_sf) = \mu(f)$ (respectively, $\mu(L_sf) = \mu(f)$) for all $s \in S$ and $f \in A$. 
Remark 1.2. It follows from [1.2.1.2] that Definition 1.1 will reduce to the definition of a scalar-valued mean when $X = \mathbb{C}$.

Of course, the evaluation mapping $\epsilon : S \to \mathcal{L}(A, X)$, defined by

$$\epsilon(s)(f) = f(s) \quad (s \in S, \ f \in A)$$

is in $M(A)$, and if $\mu \in M(A)$ and $f \in A$ is a constant function, then $\mu(f)$ is the constant.

The following proposition is obvious.

Proposition 1.3. If $A$ is a linear subspace of $\mathcal{B}(S, X)$ containing the constant functions, then each $\mu \in M(A)$ is in $\mathcal{L}(A, X)$ with $\|\mu\| = 1$.

For each $x^* \in X^*$,

$$x^*A = \{x^* f = x^* o f : f \in A\}$$

is a linear subspace of $\mathcal{B}(S)$.

Here we have adopted the definition in [2] of a mean on $A$. [3] gives a definition of a mean in terms of a scalar-valued mean on $\mathcal{S}p(X^* o A) = \mathcal{S}p(x^* A : x^* \in X^*)$. In the next lemma, we set up a connection like this, and we will show in Theorem 1.7 that the definitions of a mean in [2] and [3] are equivalent. We will deal with other applications in §4.

Lemma 1.4. Let $A$ be a linear subspace of $\mathcal{B}(S, X)$. A mapping $\mu : A \to X$ is in $M(A)$ if and only if, for each $x^* \in X^*$, there is a $\varphi_{\mu, x^*} \in M(x^* A)$ such that

$$x^* \mu(f) = \varphi_{\mu, x^*}(x^* f) \quad (f \in A).$$

If $A$ is right (left) translation invariant, then $\mu$ is right (left) invariant if and only if the $\varphi_{\mu, x^*}$'s are right (left) invariant. Furthermore, the set $\varphi_{\mu} = \{\varphi_{\mu, x^*} : x^* \in X^*\}$ is uniquely determined by $\mu$, i.e., $\varphi_{\mu, x^*} = \varphi_{\mu', x^*}$ for all $x^* \in X^*$ if and only if $\mu = \mu'$.

Proof. Sufficiency. First, $\mu$ is a linear mapping from $A$ to $X$. In fact, for $f, g \in A$ and $\alpha, \beta \in \mathbb{C}$,

$$x^* \mu(\alpha f + \beta g) = \varphi_{\mu, x^*}(x^*(\alpha f + \beta g))$$

$$= \varphi_{\mu, x^*}(x^*(\alpha f)) + \varphi_{\mu, x^*}(x^*(\beta g))$$

$$= \alpha \varphi_{\mu, x^*}(x^* f) + \beta \varphi_{\mu, x^*}(x^* g)$$

$$= \alpha x^* \mu(f) + \beta x^* \mu(g)$$

$$= x^* (\alpha \mu(f) + \beta \mu(g)).$$

The equality is true for all $x^* \in X^*$, therefore

$$\mu(\alpha f + \beta g) = \alpha \mu(f) + \beta \mu(g).$$

We claim that $\mu(f) \in \overline{co} f(S)$, for all $f \in A$. If it is not true for some $f \in A$, by the Hahn–Banach theorem there is an $x^* \in X^*$ such that
\[ |x^* \mu(f)| > \sup_{s \in S} |x^* f(s)| = \|x^* f\|. \]

It follows from Remark 1.2 and Proposition 1.3 that \( \varphi_{\mu, x^*} \in M(x^* A) \) is in \((x^* A)^*\) with \( \|\varphi_{\mu, x^*}\| = 1 \). So
\[ |x^* \mu(f)| = |\varphi_{\mu, x^*}(x^* f)| \leq \|x^* f\|, \]

a contradiction.

Necessity. For each \( x^* \in X^* \), define \( \varphi_{\mu, x^*} \in (x^* A)^* \) by
\[ \varphi_{\mu, x^*}(x^* f) = x^* \mu(f) \quad (f \in A). \]

\( \varphi_{\mu, x^*} \) is well-defined on \( x^* A \). For, if \( x^* f = 0 \) for some \( f \in A \), then \( f(S) \subset N(x^*) \), the null subspace of \( x^* \), so \( \varphi_{\mu, x^*}(x^* f) = x^* \mu(f) = 0 \) since \( \mu(f) \in \overline{\partial f}(S) \) (Definition 1.1). Clearly \( \varphi_{\mu, x^*} \) is linear on \( x^* A \). Furthermore
\[ \varphi_{\mu, x^*}(x^* f) = x^* \mu(f) \in x^* \overline{\partial f}(S) \subset \overline{\partial x^* f}(S), \]

so \( \varphi_{\mu, x^*} \) is in \( M(x^* A) \).

The rest of the lemma is clear.

We can furnish \( \mathcal{L}(A, X) \) with two topologies, both of which make \( \mathcal{L}(A, X) \) a locally convex topological space. One is the strong operator topology \( \tau_s \), which is the weakest topology of \( \mathcal{L}(A, X) \) relative to which the mapping \( U \rightarrow Uf : \mathcal{L}(A, X) \rightarrow X \) is continuous for each \( f \in A \), and the other is the weak operator topology \( \tau_w \), which is the weakest topology of \( \mathcal{L}(A, X) \) relative to which the mapping \( U \rightarrow x^* Uf : \mathcal{L}(A, X) \rightarrow C \) is continuous for each \( f \in A \) and \( x^* \in X^* \). These topologies can be relativized to \( M(A) \subset \mathcal{L}(A, X) \).

**Proposition 1.5.** Let \( A \) be a linear subspace of \( B(S, X) \). Then, for \( \tau_s \)

1. \( M(A) \) is convex and closed in \( \mathcal{L}(A, X) \);
2. \( \text{co}(\epsilon(S)) \) is dense in \( M(A) \);
3. if \( S \) is a topological space and \( A \subset C(S, X) \), then \( \epsilon : S \rightarrow M(A) \) is continuous.

Furthermore, if the range \( f(S) \) of \( f \) is relatively compact in \( X \) for each \( f \in A \), then \( M(A) \) is \( \tau_s \)-compact.

**Proof.**

1. The convexity of \( M(A) \) follows directly from Definition 1.1. To show that \( M(A) \) is closed, let \( \{\mu_\alpha\} \subset M(A) \) converge to \( \mu \in \mathcal{L}(A, X) \) for \( \tau_s \). Then \( \mu_\alpha(f) \rightarrow \mu(f) \) for each \( f \in A \), and since \( \mu_\alpha(f) \in \overline{\partial f}(S) \) for all \( \alpha \), \( \mu(f) \in \overline{\partial f}(S) \). Therefore, \( \mu \in M(A) \).
2. Clearly, \( \text{co}(\epsilon(S)) \subset M(A) \). If there is a \( \mu \in M(A) \) such that \( \mu \notin \overline{\partial}(\epsilon(S)) \), the closure being taken in \( \tau_s \), then there is an \( f \in A \) such that \( \mu(f) \notin \overline{\partial}(\epsilon(S)f) = \overline{\partial f}(S) \), which contradicts Definition 1.1.
3. is obvious.
The proof of the compactness of $M(\mathcal{A})$, if $\mathcal{A}$ satisfies the compactness condition, is similar to that of its counterpart in the following proposition, so we omit it.

**Proposition 1.6.** Let $\mathcal{A}$ be a linear subspace of $\mathcal{B}(S, X)$. Then the conclusions (1)-(3) of the previous proposition are true for $\tau_w$. Furthermore, if $\mathcal{A}$ is such that the range $f(S)$ of $f$ is weakly relatively compact in $X$ for each $f \in \mathcal{A}$, then $M(\mathcal{A})$ is $\tau_w$-compact.

**Proof.** Using Lemma 1.4, we can prove (1)-(3) in much the same way that (1)-(3) of Proposition 1.5 we proved.

We now show that $M(\mathcal{A})$ is $\tau_w$-compact when $\mathcal{A}$ satisfies the weak compactness condition. For each $x^* \in X^*$, $M(x^*\mathcal{A})$ is weak* compact [1, 2.1.8]. Therefore, the product space

$$
\prod := \prod \{M(x^*\mathcal{A}) : x^* \in X^*\}
$$

is compact in the product topology.

By Lemma 1.4, the mapping $\mu \to \varphi_\mu = \{\varphi_{\mu, x^*} : x^* \in X^*\} : M(\mathcal{A}) \to \prod$ is 1-1, and it is homeomorphism when $M(\mathcal{A})$ has the topology $\tau_w$. To show that $M(\mathcal{A})$ is $\tau_w$-compact, it suffices to show that the image of $M(\mathcal{A})$ in $\prod$ is closed.

Let $\varphi = \{\varphi_{x^*} : x^* \in X^*\} \in \prod$ and let the image $\{\varphi_{\mu_a}\}$ of $\{\mu_a\}$ converge to $\varphi$ in $\prod$. We show that there is a $\mu \in M(\mathcal{A})$ such that $\varphi$ is the image of $\mu$ and $\mu_a \to \mu$ in $\tau_w$.

Since $f(S)$ is weakly relatively compact in $X$ for each $f \in \mathcal{A}$, by the Krein-Smulian theorem [1, A.10] $\overline{\partial} f(S)$ is weakly compact in $X$ for each $f \in \mathcal{A}$. Since $\mu_a(f) \in \overline{\partial} f(S)$ for all $\alpha$ and $x^*\mu_a(f) \to \varphi_{x^*}(x^* f)$ for all $x^* \in X^*$, there is a $\mu(f) \in \overline{\partial} f(S)$ such that $x^* \mu(f) = \varphi_{x^*}(x^* f)$ for all $x^* \in X^*$. The map $f \to \mu(f)$ is clearly linear, so $\mu \in M(\mathcal{A})$. Thus $\mu_a \to \mu$ in $\tau_w$, and the proof is complete.

The following theorem shows that the definition of a mean in [2] is equivalent to that in [3].

**Theorem 1.7.** A mapping $\mu : \mathcal{A} \to X$ is in $M(\mathcal{A})$ if and only if there is a unique $\varphi_\mu \in M(\overline{\partial}(X^* \circ \mathcal{A}))$ such that

$$
x^* \mu(f) = \varphi_\mu(x^* f) \quad (x^* \in X^*, \ f \in \mathcal{A}). \quad (1.1)
$$

**Proof.** The sufficiency comes from the sufficiency in the first statement of Lemma 1.4.

Necessity. By Lemma 1.4, if $\mu$ is in $M(\mathcal{A})$, then for each $x^* \in X^*$ there is a $\varphi_{\mu, x^*}$ in $M(x^*\mathcal{A})$ such that

$$
x^* \mu(f) = \varphi_{\mu, x^*}(x^* f) \quad (f \in \mathcal{A}).
$$

We show first that $\varphi_{\mu, x^*}$ is independent of $x^* \in X^*$, i.e., if $x^*_1, x^*_2 \in X^*$ and $f_1, f_2 \in \mathcal{A}$ are such that $x^*_1 f_1 = x^*_2 f_2$, then $\varphi_{\mu, x^*_1}(x^*_1 f_1) = \varphi_{\mu, x^*_2}(x^*_2 f_2)$.

Since $\mu \in M(\mathcal{A})$, by Proposition 1.6 (2) there is a net $\{\sum_{s \in S} \lambda_s(s)\epsilon(s)\}$ converging to $\mu$ for $\tau_w$; here each $\lambda_s : S \to [0, 1]$ has finite support and satisfies $\sum_{s \in S} \lambda_s(s) = 1$. Next, $x^*_1(\sum_{s \in S} \lambda_s(s)\epsilon_1(s)) = x^*_2(\sum_{s \in S} \lambda_s(s)\epsilon_2(s))$ because $x^*_1 f_1 = x^*_2 f_2$, so
Therefore we can define $\varphi_\mu$ for $\sum_{i=1}^{m} \alpha_i x_i^* f_i \in \text{sp}(X^* \circ \mathcal{A})$ by

$$
\varphi_\mu \left( \sum_{i=1}^{m} \alpha_i x_i^* f_i \right) = \sum_{i=1}^{m} \alpha_i \varphi_{\mu,x_i}(x_i^* f_i).
$$

It is easy to see that $\varphi_\mu$ is in $M(\text{sp}(X^* \circ \mathcal{A}))$. Therefore $\varphi_\mu$ has a unique extension to $\varphi_\mu(\mathcal{A})$. The uniqueness is clear. The proof is finished.

By Theorem 1.7, we can write $\varphi_\mu$ for $\varphi_{\mu,x}$ in Lemma 1.4.

### §2. Introversion and Semigroups of Vector–Valued Means

**Definition 2.1.** Let $\mathcal{A}$ be a translation invariant linear subspace of $B(S, X)$. For a linear map $\mu$ from $\mathcal{A}$ to $X$, define the left introversion operator $T_\mu : \mathcal{A} \to B(S, X)$ by

$$
T_\mu f(s) = \mu(L_s f) \quad (f \in \mathcal{A}, \ s \in S)
$$

and analogously define the right introversion operator $U_\mu : \mathcal{A} \to B(S, X)$ by

$$
U_\mu f(s) = \mu(R_s f) \quad (f \in \mathcal{A}, \ s \in S).
$$

If $T_\mu \mathcal{A} \subset \mathcal{A}$ for all $\mu \in M(\mathcal{A})$, we will say that $\mathcal{A}$ is left introverted; we will say that $\mathcal{A}$ is right introverted if $U_\mu \mathcal{A} \subset \mathcal{A}$. $\mathcal{A}$ is introverted if it is both left and right introverted.

A semitopological semigroup $S$ is a semigroup and a Hausdorff topological space in such a way that multiplication is separately continuous, i.e., the maps $s \mapsto ts$ and $s \mapsto st$ from $S$ into $S$ are continuous for all $t \in S$. $C(S, X)$ denotes the Banach space of all continuous members of $B(S, X)$.

**Example 2.2.** $C(S, X)$ is introverted if $S$ is a compact semitopological semigroup.

For $\mu \in M(C(S, X))$ and $f \in C(S, X)$, we must show that $T_\mu f$ and $U_\mu f$ are continuous.

Let $g \in C(S)$ and let $x \in X$. $g(\cdot)x \in C(S, X)$. Theorem 1.7 implies that $\mu(g(\cdot)x) = \varphi_\mu(g)x$ and $T_\mu(g(\cdot)x) = T_{\varphi_\mu}(g)x$. Therefore $T_{\varphi_\mu}(g) \in C(S, X)$ since $T_{\varphi_\mu}(g) \in C(S)$ [1, 2.2.5]. Note the fact that $C(S, X) = \varphi_\mu \{g(\cdot)x : g \in C(S), \ x \in X\}$ since $S$ is compact. For $\epsilon > 0$ there is $p(\cdot) = \sum_{i=1}^{n} f_i(\cdot)x_i$, where $f_i \in C(S)$ and $x_i \in X$, $i = 1, 2, \cdots, n$, such that

$$
\|f - p\| < \epsilon.
$$

Now $p \in C(S, X)$ and

$$
\|T_\mu f - T_\mu p\| = \max_{s \in S} \|\mu(L_s(f - p))\| \leq \|f - p\| < \epsilon.
$$

Therefore $T_\mu f \in C(S, X)$.

Similarly $U_\mu f \in C(S, X)$. The proof is finished.
Proposition 2.3. Let $\mathcal{A}$ be a translation invariant linear subspace of $\mathcal{B}(S, X)$ containing the constant functions and let $\epsilon : S \rightarrow M(\mathcal{A})$ be the evaluation mapping. Then

1. for each $\mu \in M(\mathcal{A})$, $T_\mu : \mathcal{A} \rightarrow \mathcal{B}(S, X)$ is a bounded linear transformation with $\|T_\mu\| \leq \|\mu\|$;
2. the mapping $\mu \mapsto T_\mu : M(\mathcal{A}) \rightarrow \mathcal{L}(\mathcal{A}, \mathcal{B}(S, X))$ is a bounded transformation;
3. if $\mu \in M(\mathcal{A})$, then $T_\mu(x) = x$, $x \in X$;
4. for all $s \in S$ and $\mu \in M(\mathcal{A})$

$$T_\mu L_s = L_s T_\mu,$$

$$T_\mu R_s = T_{R_s^* \mu},$$

$$T_{\epsilon(s)} = R_s,$$

where $R_s^* : M(\mathcal{A}) \rightarrow M(\mathcal{A})$ is the adjoint of $R_s$;
5. if $f \in \mathcal{A}$, then $\{T_\mu f : \mu \in M(\mathcal{A})\}$ is the closure in $\mathcal{B}(S, X)$ of $\text{co}(R_s f)$ in the topology of pointwise convergence on $S$.

The proof of the proposition above is like that for [1, 2.2.3], so we omit it.

Definition 2.4. Let $\mathcal{A}$ be a translation invariant linear subspace of $\mathcal{B}(S, X)$ containing the constant functions, and define

$$Z_T = \{ \nu \in \mathcal{L}(\mathcal{A}, X) : T_\nu \mathcal{A} \subset \mathcal{A} \}$$

and

$$Z_U = \{ \mu \in \mathcal{L}(\mathcal{A}, X) : U_\mu \mathcal{A} \subset \mathcal{A} \}.$$ 

If $\mu \in \mathcal{L}(\mathcal{A}, X)$ and $\nu \in Z_T$, define $\mu \nu : \mathcal{A} \rightarrow X$ by

$$\mu \nu(f) = \mu(T_\nu f) \quad (f \in \mathcal{A}).$$

If $\mu \in Z_U$ and $\nu \in \mathcal{L}(\mathcal{A}, X)$, define $\mu * \nu : \mathcal{A} \rightarrow X$ by

$$\mu * \nu(f) = \nu(U_\mu f) \quad (f \in \mathcal{A}).$$

Definition 2.5. An admissible subspace $\mathcal{A}$ of $\mathcal{B}(S, X)$ is a norm closed, translation invariant, left introverted subspace of $\mathcal{B}(S, X)$ containing the constant functions. In the case that $X = \mathbb{C}$, an admissible subspace $\mathcal{A} \subset \mathcal{B}(S)$ is also required to be conjugate closed.

Let $S$ be a semigroup. Define $\rho_t : S \rightarrow S$ and $\lambda_t : S \rightarrow S$ by

$$\rho_t = st, \quad \lambda_t = ts \quad (s \in S).$$

$S$ is called a right topological semigroup if it is a topological space and $\rho_t$ is continuous for all $t \in S$. Set

$$\Lambda(S) = \{ s \in S : \lambda_s \text{ is continuous} \}.$$ 

An affine semigroup $S$ is a semigroup and a convex subset of a vector space in such a way that $\rho_t$ and $\lambda_t$ are affine mappings for each $t \in S$. The requirement that $\rho_t$ and $\lambda_t$ be affine means that if $r, s \in S$ and $a, b \in [0, 1]$ with $a + b = 1$ then

$$(ar + bs)t = art + bat \quad \text{and} \quad t(ar + bs) = atr + bts,$$
where (+) denotes vector addition.

The following lemma summarizes the properties of the operation \((\mu, \nu) \rightarrow \mu \nu\). The proof is similar to that of [1, 2.2.9]. We omit the statements of the corresponding properties of the operation \((\mu, \nu) \rightarrow \mu * \nu\).

**Lemma 2.6.** Let \(A\) be as in Definition 2.4 and let \(\epsilon : A \rightarrow X\) be the evaluation mapping. Then

1. \(Z_T\) is a linear subspace of \(\mathcal{L}(A, X)\) containing \(\epsilon(S)\);
2. \(\mu \nu \in \mathcal{L}(A, X)\) for all \(\mu \in \mathcal{L}(A, X)\) and \(\nu \in Z_T\);
3. if \(\mu \in \mathcal{L}(A, X)\), \(\nu \in Z_T\) and \(s \in S\), we have
   \[
   T_{\mu \nu} = T_{\mu} \circ T_{\nu},
   \]
   \[
   \epsilon(s) \nu = L_s^* \nu, \]
   \[
   \mu \epsilon(s) = R_s^* \mu, \quad \text{and}
   \]
   \[
   \|\mu \nu\| \leq \|\mu\| \|\nu\|,
   \]
   where \(L_s^* : M(A) \rightarrow M(A)\) is the adjoint of \(L_s\);
4. \(Z_T\) is a right topological semigroup.

The following result is essentially a consequence of the preceding lemma and Propositions 1.5 and 1.6.

**Theorem 2.7.**

1. If \(A\) is an admissible subspace of \(B(S, X)\), then for \(\tau_s\) or \(\tau_w\), and multiplication \((\mu, \nu) \rightarrow \mu \nu\), \(M(A)\) is a right topological affine subsemigroup of \(\mathcal{L}(A, X)\), \(\co(\epsilon(S)) \subset \Lambda(M(A))\) and \(\epsilon : S \rightarrow M(A)\) is a homomorphism.
2. If we also assume that \(f(S)\) is (weakly) relatively compact for all \(f \in A\), then \(M(A)\) is also compact for \((\tau_w) \tau_s\).

Let \(S\) be a compact semitopological semigroup. By Example 2.2, \(C(S, X)\) is introverted. Hence \(\mu \nu, \mu * \nu \in M(C(S, X))\); indeed, they are equal.

**Proposition 2.8.** Let \(S\) be a compact semitopological semigroup and let \(A = C(S, X)\). Then

1. \(\mu \nu = \mu * \nu\) for all \(\mu, \nu \in M(A)\);
2. for \(\tau_s\) and multiplication \((\mu, \nu) \rightarrow \mu \nu\), \(M(A)\) is a compact semitopological affine semigroup;
3. if \(S\) is also a topological semigroup, so is \(M(A)\) in \(\tau_s\).

**Proof.** (1). Note that \(\varphi_\mu \varphi_\nu = \varphi_{\mu * \nu}\) [1, 2.2.12 (a)]. Similar to the proof of Example 2.2, we have, for \(g \in C(S)\) and \(x \in X\),

\[
\mu \nu(g(\cdot) x) = \mu(T_\nu g(\cdot) x) = \varphi_\mu(T_{\varphi_\nu} g) x = \varphi_\mu \varphi_\nu(g) x = \mu \nu(g(\cdot) x).
\]
Therefore
\[ \mu \nu(f) = \mu \ast \nu(f) \quad (f \in C(S, X)), \]
i.e., \( \mu \nu = \mu \ast \nu \).

(2) is a consequence of (1) and Theorem 2.7 (1).

To verify (3), we need to show that if \( \mu_\alpha \to \mu \) and \( \nu_\alpha \to \nu \) for \( \tau \) then \( \mu_\alpha \nu_\alpha \to \mu \nu \) for \( \tau \). Note that \( \varphi_{\mu_\alpha} \varphi_{\nu_\alpha}(g) \to \varphi_{\mu} \varphi_{\nu}(g) \) for every \( g \in C(S) \) [1, 2.2.12 (c)]. Now, for \( x \in X \),
\[ \mu_\alpha \nu_\alpha(g(\cdot)x) = \varphi_{\mu_\alpha} \varphi_{\nu_\alpha}(g)x \to \varphi_{\mu} \varphi_{\nu}(g)x = \mu \nu(g(\cdot)x). \]
Again using the fact that \( C(S, X) = \overline{p}(g(\cdot)x : g \in C(S), x \in X) \), we have \( \mu_\alpha \nu_\alpha(f) \to \mu \nu(f) \) for every \( f \in C(S, X) \).

§3. Invariant Vector-Valued Means

\( S \) denotes a semigroup which need not have an identity and \( A \) denotes a linear subspace of \( B(S, X) \) containing the constant functions. Let \( \text{LIM}(A) \) (\( \text{RIM}(A) \)) denotes the set of left (right) invariant means on \( A \). \( A \) is said to be left (right) amenable if \( \text{LIM}(A) \neq \phi \) \( (\text{RIM}(A) \neq \phi) \). If \( A \) is translation invariant, we set
\[ IM(A) = \text{LIM}(A) \cap \text{RIM}(A) \]
and call members of \( IM(A) \) invariant means. \( A \) is said to be amenable if \( IM(A) \neq \phi \).

As in the scalar case, we have the following proposition, whose proof is similar to that of [1, 2.3.5]; so we omit it.

Proposition 3.1. Let \( A \) be an admissible subspace of \( B(S, X) \) and let \( \epsilon : S \to \mathcal{L}(A, X) \) be the evaluation mapping.

(1) \( \text{LIM}(A) \) is the set of right zeros of \( M(A) \); hence if \( A \) is left amenable, then \( \text{LIM}(A) \) is a closed ideal of \( M(A) \) contained in every right ideal.

(2) If \( A \) is right amenable, then \( \text{RIM}(A) \) is a closed left ideal of \( M(A) \).

Corollary 3.2. Let \( A \) be an admissible subspace of \( B(S, X) \). If \( A \) is left and right amenable, then it is amenable.

Proof. If \( \mu \in \text{LIM}(A) \) and \( \nu \in \text{RIM}(A) \), then \( \mu \nu \in IM(A) \).

Corollary 3.3. Let \( A \) be an admissible right introverted subspace of \( B(S, X) \) such that \( \mu \nu = \mu \ast \nu \) for all \( \mu, \nu \in M(A) \). Then \( A \) has at most one invariant mean.

Proof. By the proposition and its right introverted analog, if \( \mu, \nu \in IM(A) \), then \( \nu = \mu \nu = \mu \ast \nu = \mu \).

Theorem 3.4. Let \( A \) be an admissible subspace of \( B(S, X) \) such that, for each \( f \in A \), the range \( f(S) \) of \( f \) is relatively weakly compact. Let \( K(f) \) denote the closure in \( B(S, X) \) of \( co(R_S f) \) for the pointwise topology. The following assertions are equivalent:
(1) $\mathcal{A}$ is left amenable:

(2) for each $f \in \mathcal{A}$, $K(f)$ contains a constant function;

(3) for each $f \in \mathcal{A}$ and $s \in S$, $0 \in K(f - L_sf)$.

Furthermore, if (1) holds then, for each $f \in \mathcal{A}$, $\{\mu(f) : \mu \in \text{LIM}(\mathcal{A})\}$ is the set of constant functions in $K(f)$.

Proof. We omit the proofs that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) which do not use weak compactness hypothesis. Here we show that (3) $\Rightarrow$ (1).

For each $f \in \mathcal{A}$ and $s \in S$, let

$$M(f, s) = \{\mu \in M(\mathcal{A}) : T_{\mu}(f - L_sf) = 0\}.$$ 

The sets $M(f, s)$ are $\tau_w$-closed, and therefore $\tau_w$-compact. For, let $\{\mu_\alpha\} \subset M(f, s)$ converge to $\mu \in M(\mathcal{A})$. We want to show that $\mu \in M(f, s)$, i.e.,

$$T_{\mu}(f - L_sf) = 0.$$

Note that

$$T_{\mu}(f - L_sf)(t) = \mu(L_tf - L_tsf) \quad (t \in S)$$

and $\mu_\alpha(L_tf - L_tsf) = T_{\mu_\alpha}(f - L_sf)(t) = 0$ for all $\alpha$. Since $\mu_\alpha(L_tf - L_tsf) \to \mu(L_tf - L_tsf)$ weakly, $\mu(L_tf - L_tsf) = 0$. That is, $T_{\mu}(f - L_sf) = 0$.

As in the proof of [1, 2.3.11], we can show that the family $\{M(f, s) : f \in \mathcal{A}, s \in S\}$ has the finite intersection property. By Proposition 1.6 $M(\mathcal{A})$ is $\tau_w$-compact. So

$$\bigcap\{M(f, s) : f \in \mathcal{A}, s \in S\} \neq \emptyset.$$

Let $\mu$ be any member of this intersection, then $\mu^2 \in \text{LIM}(\mathcal{A})$.

Let $S$ be a group and let $\mathcal{A}$ be a linear subspace of $B(S, X)$. For each $f \in \mathcal{A}$ define $\tilde{f} : S \to X$ by

$$\tilde{f}(s) = f(s^{-1}) \quad (s \in S),$$

and set

$$\hat{\mathcal{A}} = \{\tilde{f} : f \in \mathcal{A}\}.$$

If $\mu \in M(\mathcal{A})$, define $\tilde{\mu} \in M(\hat{\mathcal{A}})$ by

$$\tilde{\mu}(\tilde{f}) = \mu(f) \quad (f \in \mathcal{A}).$$

If $\hat{\mathcal{A}} = \mathcal{A}$ and $\tilde{\mu} = \mu$, then $\mu$ is said to be inversion invariant.

**Theorem 3.5.** Let $G$ be a compact Hausdorff topological group. Then $C(G, X)$ has a unique invariant mean $\mu$. Furthermore $\mu$ is inversion invariant.

**Proof.** The mean $\mu$ can be expressed as

$$\mu(f) = \int_G f d\nu \quad (f \in C(G, X)),$$

where $\nu$ is normalized Haar measure on $G$; the properties of $\mu$ follows from those of $\nu$. 
The scalar version of the next theorem is [1, 2.3.14]; a similar result has appeared in [3], but there $S$ is required to have an identity. A small modification of the proof of [1, 2.3.14] yields a proof of the present theorem.

**Theorem 3.6.** Let $S$ be a compact Hausdorff semitopological semigroup. Then the following assertions hold:

1. $\mathcal{C}(S, X)$ is left (respectively right) amenable if and only if $S$ has a unique minimal right (respectively, left) ideal;
2. $\mathcal{C}(S, X)$ is amenable if and only if the minimal ideal of $S$ is a compact topological group.

§4. Vector–Valued Weakly Almost Periodic Functions

Let $S$ be a semitopological semigroup; we do not assume $S$ has an identity. Let $\mathcal{WAP}(S, X)$ consist of those members $f$ of $\mathcal{C}(S, X)$ for which the right orbit $R_{S}f = \{R_{s}f : s \in S\}$ is weakly relatively compact in $\mathcal{C}(S, X)$.

With a proof similar to that for [1, 4.2.5], one sees that the space $\mathcal{WAP}(S, X)$ is a closed translation invariant subspace of $\mathcal{C}(S, X)$. When $X = \mathbb{C}$, $\mathcal{WAP}(S, X)$ is just $\mathcal{WAP}(S)$, the $C^*$-algebra of weakly almost periodic functions on $S$. We note that

$$x^* \circ \mathcal{WAP}(S, X) = \mathcal{WAP}(S) \quad (x^* \in X^*, \ x^* \neq 0).$$

Recall that $\epsilon : S \to \mathcal{L}(A, X)$ is the evaluation mapping $\epsilon(s)f = f(s), f \in \mathcal{WAP}(S, X)$. When $X = \mathbb{C}$ we denote this mapping by $\epsilon'$.

Let $aS^{\mathcal{WAP}}$ denote the $w^*$ closure in $\mathcal{WAP}(S)^*$ of $\text{coe}'(S)$; $aS^{\mathcal{WAP}}$ is a compact affine semitopological semigroup [1, 4.2.11].

**Theorem 4.1.** Let $S$ be a semitopological semigroup and let $A = \mathcal{WAP}(S, X)$. The following assertions hold:

1. $A$ is an admissible subspace of $B(S, X)$;
2. for $\tau_w$ and multiplication $(\mu, \nu) \to \mu \nu$, $M(A)$ is an affine semitopological semigroup;
3. if $f(S)$ is weakly relatively compact in $X$ for each $f \in A$, then $M(A)$ is $\tau_w$–compact;

in this case $A$ is left amenable if and only if $\mathcal{WAP}(S)$ is left amenable.

**Proof.** (1) Since $A$ is a closed translation invariant subspace of $\mathcal{C}(S, X)$, to show that $A$ is admissible we need to show that $A$ is left introverted, i.e., if $f \in A$ then $T_{\mu}f \in A$ for all $\mu \in M(A)$.

Define $V : M(A) \to B(S, X)$ by

$$V(\mu) = T_{\mu}f \quad (\mu \in M(A)).$$

By Proposition 2.3 (5)

$$V(M(A)) = \overline{\text{co}}(R_{S}f), \quad (4.1)$$
the closure being taken in the pointwise topology. Since \( f \in \mathcal{A} \), \( \text{co}(Rsf) \) is weakly relatively compact in \( \mathcal{A} \); in view of (4.1) this implies that \( V(M(\mathcal{A})) \) is the weak closure in \( \mathcal{A} \) of \( \text{co}(Rsf) \). So \( T_{\mu}f \in \mathcal{A} \) for all \( \mu \in M(\mathcal{A}) \).

(2) By Theorem 2.7 (1), for \( \tau_w \) and multiplication \( (\mu, \nu) \to \mu\nu \), \( M(\mathcal{A}) \) is a right topological affine semigroup. It follows from Theorem 1.7 that the mapping \( \Pi : \mu \to \varphi_\mu \) is a \( \tau_w - \text{w}^* \) homeomorphism of \( M(\mathcal{A}) \) into \( aS^{\text{WAP}} \). Since \( x^*\nu(f) = \varphi_\nu(x^*f) \) for \( f \in \mathcal{A} \) and \( x^* \in X^* \), \( x^*(T_{\nu}f) = T_{\varphi_\nu}(x^*f) \). It follows that \( \varphi_{\mu\nu} = \varphi_\mu \varphi_\nu \). Since \( \Pi(\mu\nu) = \varphi_{\mu\nu} \), \( \Pi \) is a homomorphism too. So \( M(\mathcal{A}) \) is an affine semitopological semigroup because \( aS^{\text{WAP}} \) is.

(3) When \( \mathcal{A} \) satisfies the compactness condition, the \( \tau_w \)-compactness of \( M(\mathcal{A}) \) is a consequence of Theorem 2.7 (2). In this case, \( M(\mathcal{A}) \cong aS^{\text{WAP}} \). So we get the last statement.

The proof is complete.

**Remark 4.2.** For \( f \in \mathcal{WAP}(S, X) \), in general \( f(S) \subset X \) is not weakly relatively compact. However, if \( S \) admits an identity, it follows from the double limit property (e.g., [2, Theorem 3]) that \( f(S) \) is weakly relatively compact. Of course, if \( X \) is reflexive then \( f(S) \) is weakly relatively compact.

**Theorem 4.3.** For a compact semitopological semigroup \( S \), \( \mathcal{WAP}(S, X) \subset C(S, X) \).

The theorem holds because the facts of \( C(S, X) = \overline{\text{co}} \{ f(\cdot) : f \in C(S), x \in X \} \) and \( \mathcal{WAP}(S) = C(S) \) [1, 4.2.9].

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**References**