A CLASS OF BOUNDED STARLIKE FUNCTIONS

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ABSTRACT. We consider functions \( f(z) = z + \cdots \) that are analytic in the unit disk and satisfy there the inequality \( \text{Re} (f'(z) + zf''(z)) > \alpha, \alpha < 1 \). We find extreme points and then determine sharp lower bounds on \( \text{Re} f'(z) \) and \( \text{Re} (f(z)/z) \). Sharp results for the sequence of partial sums are also found.

KEY WORDS AND PHRASES. Univalent, starlike.

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1. INTRODUCTION.

Denote by \( A \) the family of functions \( f(z) = z + \cdots \) that are analytic in the unit disk \( \Delta = \{ z : |z| < 1 \} \) and by \( S \) the subfamily of functions that are univalent in \( \Delta \). Let \( R \) be the functions \( f \) in \( A \) for which \( \text{Re}(f'(z) + zf''(z)) > 0, z \in \Delta \). Chichra [1] showed that \( R \subseteq S \). In fact, he proved that \( \text{Re} f'(z) > 0, z \in \Delta \), and hence \( R \subseteq C \), the class of close-to-convex functions. R. Singh and S. Singh [4] showed that \( R \subseteq S^* \), the family of starlike functions. They later found in [5] for \( f \in R \) and \( z \in \Delta \) that \( \text{Re}(f(z)/z) > 1/2 \) and that the partial sums \( S_n(z, f) \) satisfy \( \text{Re}(S_n(z, f)/z) > 1/3 \). Neither of these results is sharp.

In this note, we find the sharp bounds. Our results will be put into a slightly more general context. Denote by \( R(\alpha), \alpha < 1 \), the subfamily of \( A \) consisting of functions \( f \) for which \( \text{Re}(f'(z) + zf''(z)) > \alpha, z \in \Delta \). Denote by \( P(\alpha), \alpha < 1 \), the subfamily of \( A \) consisting of functions \( f \) for which \( \text{Re} f'(z) > \alpha, z \in \Delta \). It was shown in [5] that \( R(\alpha) \subseteq S^* \) for \( \alpha \geq -1/4 \). We improve this lower bound and also find the smallest \( \alpha \) for which \( R(\alpha) \subseteq S \). Our approach in this note will be to characterize the extreme points of \( R(\alpha) \), which lead to sharp bounds for certain linear problems.

2. MAIN RESULTS.

THEOREM 1. (i) The extreme points of \( R(\alpha) \) are

\[
\{ (2\alpha - 1)z + (2\alpha - 2)x \log(1 - xt) \}_{t = 0}^1, \quad |x| = 1.
\]

(ii) A function \( f \) is in \( R(\alpha) \) if and only if \( f \) can be expressed as

\[
F(z) = \int_X f(z) \, d\mu(x),
\]

where \( \mu \) varies over the probability measures defined on the unit circle \( X \).

PROOF of (i). Hallenbeck [2] showed that the extreme points of \( P(\alpha) \) are

\[
\{ (2\alpha - 1)z + (2\alpha - 2)x \log(1 - xz), \quad |x| = 1 \}. \tag{2.1}
\]
Since \( (zf')' = f' + zf'' \), we have \( f \in R(\alpha) \) if and only if \( zf' \in P(\alpha) \). Hence the operator \( L \) defined by \( L(f) = \int_0^x (f(t)/t)dt \) is a linear homeomorphism \( L:P(\alpha) \to R(\alpha) \) and thus preserves extreme points.

**Proof of (ii).** The family \( R(\alpha) \) is convex and is therefore equal to its convex hull. This enables us to characterize \( f \in R(\alpha) \) by 

\[
F(z) = \int_0^x f_x(z)\,d\mu(z).
\]

**Corollary 1.** If \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in R(\alpha) \), then \( |a_n| \leq 2(1-\alpha)/n^2 \).

The result is sharp.

**Proof.** The coefficient bounds are maximized at an extreme point. Now \( f_x(z) \) may be expressed as

\[
f_x(z) = z + 2(1-\alpha) \sum_{n=2}^{\infty} \frac{x^{n-1}z^n}{n^2} , \quad |x| = 1 , \quad (2.2)
\]

and the result follows.

**Corollary 2.** If \( f \in R(\alpha) \), then \( |f(z)| \leq (1-\alpha) \left( \frac{\pi^2}{3} - 1 \right) + \alpha \).

**Proof.** From (2.2), we see that \( |f(z)| \leq r + 2(1-\alpha) \sum_{n=2}^{\infty} r^n , \quad |z| = r \). Letting \( r \to 1 \), we get

\[
|f(z)| \leq 1 + 2(1-\alpha) \left( \frac{\pi^2}{3} - 1 \right) = (1-\alpha) \left( \frac{\pi^2}{3} - 1 \right) + \alpha .
\]

Corollary 2 shows that the family \( R(\alpha) \) is bounded in \( \Delta \) for all real \( \alpha, \alpha < 1 \), even though its functions may not be univalent. Note from (2.1) that the extreme points of \( P(\alpha) \) are unbounded in \( \Delta \) for all \( \alpha < 1 \).

In the next two theorems, we will be looking at continuous linear operators \( L(f) = Re f' \) and \( L(f) = Re f(z)/z \) acting on \( R(\alpha) \). It therefore suffices to investigate the extreme points in determining minima. Since \( R(\alpha) \) is rotationally invariant, we may restrict our attention to the extreme point

\[
g(z) = (2\alpha - 1)z - 2(1-\alpha) \int_0^1 \frac{\log(1-t)}{t} \, dt = z + 2(1-\alpha) \sum_{n=2}^{\infty} \frac{\pi^n}{n} , \quad (2.3)
\]

**Theorem 2.** If \( f \in R(\alpha) \), then

\[
Re f'(z) > (1-\alpha)(2 \log 2 - 1) + \alpha \quad (z \in \Delta).
\]

The result is sharp.

**Proof.** We need only consider \( g(z) \) defined by (2.3). We have

\[
g'(z) = (2\alpha - 1) - 2(1-\alpha) \frac{\log(1-z)}{z} . \quad (2.4)
\]

In [2] it is shown that

\[
Re - \frac{\log(1-z)}{z} \geq \frac{\log(1+r)}{r} , \quad |z| = r , \quad (2.5)
\]

so that \( Re \, g'(z) \geq (2\alpha - 1) + 2(1-\alpha) \frac{\log(1+r)}{r} \). Letting \( r \to 1 \), the result follows.

The case \( \alpha = 0 \) is found in [5].

**Corollary 1.** \( R(\alpha) \subset S \) for \( \alpha \geq -\frac{1}{2} \left( \frac{2 \log 2 - 1}{1 - \log 2} \right) = \alpha_0 \approx -0.63 \) and \( R(\alpha) \not\subset S \) for \( \alpha < \alpha_0 \).

**Proof.** We know that \( P(0) \subset S \). Since \( (1-\alpha)(2 \log 2 - 1) + \alpha = 0 \) for \( \alpha = \alpha_0 \), the first part is a consequence of Theorem 2. The result cannot be extended to \( \alpha < \alpha_0 \) because \( g'(-1) = 0 \) at \( \alpha = \alpha_0 \). Thus \( g'(-r) = 0 \) for some \( r = r(\alpha) < 1 \) when \( \alpha < \alpha_0 \).

**Corollary 2.** \( \sum_{k=1}^{\infty} \cos k\theta \geq \sum_{k=1}^{\infty} \frac{(-1)^k}{k+1} = \log 2 - 1 \).
PROOF. From (2.3) we have
\[ \Re g'(z) = 1 + 2(1 - \alpha) \sum_{k=1}^{\infty} \frac{r^k \cos k\theta}{k+1}, \quad |z| = r, \]
which according to (2.4) and (2.5) is minimized when \( \theta = \pi \). We then let \( r \to 1 \).

In [5] it is shown that \( \Re(f(z)/z) > 1/2 \) for all \( f \) in \( R \). The next theorem improves this lower bound to \( \frac{\pi^2}{6} - 1 \approx 0.645 \). But first we state

**LEMMA 1.** \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12} \).

**Proof.** \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 2 \sum_{n=1}^{\infty} \frac{1}{(2n)^2} \), so that

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{12}. \]

**THEOREM 3.** If \( f \in R(\alpha) \), then
\[ \Re \left( \frac{f(z)}{z} \right) > (1 - \alpha) \left( \frac{\pi^2}{6} - 1 \right) + \alpha \quad (z \in \Delta). \]

The result is sharp, with the extremal function \( g \) defined by (2.2).

**Proof.** Again, we need only consider
\[ \frac{g(z)}{z} = (2\alpha - 1) - 2(1 - \alpha) \int_0^1 \frac{\log(1 - t)}{tz} \, dt. \]

Setting \( t = vz \), we may write
\[ \frac{g(z)}{z} = (2\alpha - 1) - 2(1 - \alpha) \int_0^1 \frac{\log(1 - vz)}{vz} \, dv. \]

Since \( \Re \left( - \frac{\log(1 - w)}{|w|} \right) \geq \frac{\log(1 + |w|)}{|w|}, \) \( |w| < 1 \), we get from (2.6) that for \( |z| = r \),
\[ \Re \frac{g(z)}{z} \geq (2\alpha - 1) + 2(1 - \alpha) \int_0^1 \frac{\log(1 + vr)}{vr} \, dv = \frac{g(-r)}{-r}. \]

But from (2.3) we see that
\[ \frac{g(-r)}{-r} = 1 + 2(1 - \alpha) \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2} > 1 + 2(1 - \alpha) \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)^2}. \]

An application of Lemma 1 yields
\[ \Re \frac{f(z)}{z} \geq \frac{g(-r)}{-r} > 1 + 2(1 - \alpha) \left( \frac{\pi^2}{12} - 1 \right) = \left( 1 - \alpha \right) \left( \frac{\pi^2}{6} - 1 \right) + \alpha. \]

In [5], R. Singh and S. Singh showed that \( R(\alpha) \subset S^* \) for \( \alpha \geq -1/4 \). Our sharp bound in Theorem 3 gives the following improvement.

**COROLLARY.** \( R(\alpha) \subset S^* \) for \( \alpha \geq \frac{6 - \pi^2}{24 - \pi^2} \approx -0.2738 \).

**Proof.** The result follows from Theorem 3 upon solving the inequality
\[ \alpha \geq -\frac{1}{2} \left( (1 - \alpha) \left( \frac{\pi^2}{6} - 1 \right) + \alpha \right). \]
The next lemma, due to Rogosinski and Szegő, will be needed for our results on partial sums.

**Lemma 2 [3].** \( \sum_{k=1}^{n} \frac{\cos k\theta}{k+1} \geq -\frac{1}{2} \)

**Theorem 4.** Denote by \( S_n(z,f) \) the \( n \)th partial sum of a function \( f \) in \( R(\alpha) \). If \( f \in R(\alpha) \), then

(i) \( S_n(z,f) \in P(\alpha) \),
(ii) \( \text{Re} \frac{S_n(z,f)}{z} > \frac{1+\alpha}{2}, \quad z \in \Delta \).

The results are sharp, with extremal function \( g(z) \) defined by (2.3) and \( n = 2 \).

**Proof of (i).** As before, it suffices to prove our results when \( f(z) = g(z) \). We have

\[
S'_n(z,g) = 1 + 2(1-\alpha) \sum_{k=2}^{n} \frac{z^{k-1}}{k} = 1 + 2(1-\alpha) \sum_{k=1}^{n-1} \frac{r^k \cos k\theta}{k+1}.
\]

By Lemma 2 and the minimum principle for harmonic functions,

\[
\text{Re} S'_n(z,g) > 1 + 2(1-\alpha)(-\frac{1}{2}) = \alpha
\]

**Proof of (ii).** We have

\[
\text{Re} \frac{S_n(z,g)}{z} = 1 + 2(1-\alpha) \sum_{k=1}^{n-1} \frac{r^k \cos k\theta}{(k+1)^2}.
\]

Since \( 1/(k+1) \) is decreasing, we use Lemma 2 and summation by parts to obtain

\[
\sum_{k=1}^{n-1} \left( \frac{1}{k+1} \right) \left( \frac{\cos k\theta}{k+1} \right) \geq \frac{1}{2} \left( -\frac{1}{2} \right) = -\frac{1}{4}.
\]

Substituting inequality (2.8) into (2.7) and applying the minimum principle, we get

\[
\text{Re} \frac{S_n(z,g)}{z} > 1 + 2(1-\alpha) \left( -\frac{1}{4} \right) = \frac{1+\alpha}{2}.
\]

In the special case \( \alpha = 0 \), (i) gives the result found in [5] and (ii) improves the estimate of 1/3 to the sharp bound of 1/2.

**Remark.** This work was completed while the author was a Visiting Scholar at the University of Michigan.

**References**