THE SECOND CONJUGATE ALGEBRAS OF BANACH ALGEBRAS

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ABSTRACT. In this paper, we study Arens regularity of a Banach algebra A. In particular, we give characterizations for A to be Arens regular.

KEY WORDS AND PHRASES. Banach algebra, Arens products, Arens regularity, weakly compact operators.

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1. INTRODUCTION.

Let A be a Banach algebra. It is an interesting and difficult problem to determine whether A is Arens regular. Many papers have been written on this subject. For example, see [2], [5], [9], [10], [11] and [12]. In particular let A be a B*-algebra. It is well known that A is Arens regular. However, it is not easy to prove this result. There are many different proofs of this result. For example, see [4], [5], and [8].

In this paper, we give characterizations for A to be Arens regular. It follows from this result and a result of C.A. Akemann that a B*-algebra is Arens regular. We also show that if A is a Banach algebra which is Arens regular, then any closed subalgebra of A is also Arens regular.

2. NOTATION AND PRELIMINARIES.

Definitions not explicitly given are taken from Rickart [7].

Let A be a Banach algebra and let A* and A** be the conjugate and second conjugate spaces of A. We will denote by \( r \) the canonical embedding of A into A**. The two Arens products on A** are defined in stages according to the following rules (see [3]). Let \( z, y \in A \), \( f, F \in A^* \).

Define \( loz \) by \( (loz)(y) = f(zy) \). Then \( loz \in A^* \).

Define \( Gof \) by \( (Gof)(z) = G(loz) \). Then \( Gof \in A^* \).

Define \( FoG \) by \( (FoG)(f) = F(Gof) \). Then \( FoG \in A^{**} \).

A** is a Banach algebra under the Arens product \( o_0 \), and we denote this algebra by \( (A^{**}, o_0) \).

Define \( zo'f \) by \( (zo'f)(y) = f(yz) \). Then \( zo'f \in A^* \).

Define \( fo'F \) by \( (fo'F)(x) = F(zo'f) \). Then \( fo'F \in A^* \).

Define \( Fo'G \) by \( (Fo'G)(f) = G(fo'F) \). Then \( Fo'G \in A^{**} \).

A** is a Banach algebra under the Arens product \( o' \), and we denote this algebra by \( (A^{**}, o') \).

Both of the Arens products extend the given multiplication on A when A is canonically embedded in A**. In general, \( o \) and \( o' \) are distinct on A**. If they agree on A**, then A is called Arens regular.

In this paper, all algebras and linear spaces under consideration are over the complex field \( C \).
3. ARENS REGULARITY FOR BANACH ALGEBRAS.

Let $A$ be a Banach algebra and $f \in A^*$. Define $L_f: A \to A^*$ by

$$L_f(x) = f(x), \quad (x \in A).$$

Then $L_f$ is clearly a continuous linear operator from $A$ to $A^*$. For each $F \in A^{**}$, define $F.L_f$ by

$$F.L_f(x) = (F.L_f)(x) = F(L_f(x)) = F(f(x)) = (Fof)(x).$$

Then $F.L_f \in A^*$. Define $L_f^*: A^{**} \to A^*$ by

$$L_f^*(F) = F.L_f = Fof \quad (F \in A^{**}).$$

Then $L_f^*$ is clearly a continuous linear operator from $A^{**}$ to $A^*$. For each $F \in A^{**}$, define $F.L_f^*$ by


Then $F.L_f^* \in A^{***}$. Finally, we define $L_f^*: A^{**} \to A^{***}$ by

$$L_f^*(F) = F.L_f^* \quad (F \in A^*).$$

Then clearly $L_f^*$ is a continuous linear operator from $A^{**}$ to $A^{***}$.

THEOREM 1. Let $A$ be a Banach algebra. Then the following statements are equivalent:

1. $A$ is Arens regular.
2. For each $f \in A^*$, $L_f^*(A^{**})$ is contained in $\pi(A^*)$, where $\pi(A^*)$ is a subspace of $A^{***}$.
3. For each $f \in A^*$, $L_f$ is weakly compact.
4. Let $F, G \in A^{**}$ and $\{x_\alpha\}$ a bounded net in $A$. If $\pi(x_\alpha) \to F$ weakly, then $f_\alpha F$ is a weakly limit point of $\{f_\alpha x_\alpha\}$.

PROOF. (1) $\Rightarrow$ (2). Assume (1). Let $F, G \in A^{**}$. Then $L_f^*(f) = F.L_f^*$ and by (1)

$$F.L_f^*(G) = F(L_f^*(G)) = F(Gof) = (Fof)(f) \quad (f \in A^*).$$

Therefore $F.L_f^* = \pi(f_\alpha F) \in \pi(A^*)$ and so $L_f^*(F) = F.L_f^* \in \pi(A^*)$. This proves (2).

(2) $\Rightarrow$ (3). This follows immediately from [6; p. 482, Theorem 2].

(3) $\Rightarrow$ (4). Assume that $L_f$ is weakly compact. Let $F$ and $G \in A^{**}$. Then by Goldstine's theorem [6; p. 424, Theorem 5] there exists a bounded net $\{x_\alpha\}$ in $A$ such that $\pi(x_\alpha) \to F$ weakly. Similarly, there exists a bounded net $\{y_\beta\}$ such that $\pi(y_\beta) \to G$ weakly. Since $L_f$ is weakly compact, we can assume that $L_f(x_\alpha) \to g$ weakly for some $g \in A^*$. Hence $f_\alpha G$ is a weakly limit point of $\{f_\alpha x_\alpha\}$.

Therefore $f_\alpha F$ is a weak limit point of $\{f_\alpha x_\alpha\}$. This proves (4).

(4) $\Rightarrow$ (1). Assume (4). Let $F, G \in A^{**}$. Then by Goldstine's theorem, there exists a bounded net $\{x_\alpha\}$ in $A$ such that $\pi(x_\alpha) \to F$ weakly. Since $f_\alpha F$ is a weakly limit point of $\{f_\alpha x_\alpha\}$, we can assume that

$$G(f_\alpha F) = \lim G(f_\alpha x_\alpha) = \lim Gf_\alpha(Gof)(x_\alpha) = \lim Gf_\alpha \pi(x_\alpha)(Gof) = F(Gof).$$

Therefore $f_\alpha F$ is a weak limit point of $\{f_\alpha x_\alpha\}$. This proves (4).
Therefore \((F \circ G)(f) = G(f \circ F) = F \circ G(f)\) and so \(A\) is Arens regular. This completes the proof of the theorem.

**COROLLARY 2.** Let \(A\) be a Banach algebra such that each continuous linear map \(T\) of \(A\) into \(A^*\) is weakly compact, then \(A\) is Arens regular.

**PROOF.** Since each \(L_f(f \in A^*)\) is weakly compact, \(A\) is Arens regular by Theorem 1.

Let \(A\) be a \(B^*\)-algebra and \(B\) a Banach space such that \(B^*\) is a \(W^*\)-algebra. Then by [1; p.293, Corollary II.9], any continuous linear map \(T\) of \(A\) into \(B\) is weakly compact. Therefore it follows from Corollary 1 that \(A\) is Arens regular. The property that "any continuous linear map \(T\) of \(A\) into \(B\) is weakly compact" is a very strong one. In order for \(A\) to be Arens regular, we need only to show that \(L_f\) is weakly compact for all \(f\) in \(A^*\). Therefore, a simple proof for a \(B^*\)-algebra to be Arens regular may exist.

4. **SUBALGEBRAS OF A BANACH ALGEBRA WHICH IS ARENS REGULAR.**

Let \(A\) be a Banach algebra which is Arens regular. It is well known that a subalgebra of \(A\) may not be Arens regular. In fact, let \(M\) be the group algebra of an infinite abelian locally compact group. Then \(M\) is an \(A^*\)-algebra. Let \(A\) be the completion of \(M\) in an auxiliary norm. By [5; p.857, Theorem 3.14] \(M\) is not Arens regular. Since \(A\) is a \(B^*\)-algebra, \(A\) is Arens regular.

Let \(A\) be a Banach algebra and \(M\) a closed subalgebra of \(A\). For each \(f \in A^*\), we define \(f_M\) by \(f_M(x) = f(x)\) for all \(x \in M\). Then \(f_M \in M^*\).

**THEOREM 3.** Let \(M\) be a closed subalgebra of \(A\). If \(A\) is Arens regular, then so is \(M\).

**PROOF.** Let \(f \in M^*\). Then there exists some \(f \in A^*\) such that \(f_M = f\). Let \(F \in M^{**}\). Define \(\tilde{F}\) by

\[
\tilde{F}(g) = F(g_M) \quad (g \in A^*).
\]

Then it is clear that \(\tilde{F} \in A^{**}\). Since \(A\) is Arens regular, by Theorem 1, \(L_f\) is weakly compact on \(A\). Let \(\{x_\alpha\}\) be a bounded net in \(M\), then \(L_f(x_\alpha) = f\alpha(x_\alpha) \rightharpoonup g\) weakly for some \(g \in A^*\). Since \((f\alpha(x_\alpha))_M = f\alpha(x_\alpha) \in M^*\), it follows that

\[
F(g_M) = \tilde{F}(g) = \lim_{\alpha} \tilde{F}(f\alpha(x_\alpha)) = \lim_{\alpha} F((f\alpha(x_\alpha))_M) = \lim_{\alpha} F(f\alpha(x_\alpha)).
\]

Therefore \(L_f(x_\alpha) \rightharpoonup g_M\) weakly and so by Theorem 1, \(M\) is Arens regular. This completes the proof.

**REFERENCES**


