A THEOREM OF DIFFERENTIAL MAPPINGS OF RIEMANN SURFACES

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ABSTRACT. In this paper, we have extended S.S. Chern's second basic theorem about holomorphic mapping between two Riemann surfaces to more general case, and also obtained two similar results.

KEY WORDS AND PHRASES: Riemann surfaces, differential mapping, meromorphic functions, critical points, integral form.


1. INTRODUCTION.


Let $M$ be a compact Riemann surface, $G$ be a Hermitian metric of $M$ which has constant Gauss curvature $K$. Let $\Omega$ be the volume element of $G$. For every $a \in M$, Chern has proved that there exists a real function $U_a$ which is $C^\infty$ on $M - \{a\}$, and satisfies:

$$\lambda_a = \Omega,$$

where $\lambda_a = \frac{1}{2\pi}d^c U_a$, $d^c = i(\overline{\partial} - \partial)$. If $z$ is a local coordinate function on the neighbour $U$ of $a$, such that $z(a) = 0$, then $U_a(z) + \log |z|$ is $C^\infty$ on $U$. In [5], Chern proves the following theorems.

THEOREM A. Let $D$ be a compact differentiable, orientable domain bounded by a sectiouonally smooth curve $\partial D$, $f : D \mapsto M$ is a differential mapping, if $a \in M$ such that, $f^{-1}\{a\} \cap \partial D = \emptyset$, and $f^{-1}\{a\}$ is a finite set of points, then we have:
\[ n(D,a) + \int_{\partial D} f^*\lambda_a = v(D), \]  \hspace{1cm} (2)

where \( v(D) = \frac{1}{\lambda} \int_{A} f^*\Omega \) (\( A = \int_{\partial D} \Omega \)), \( n(D,a) \) is the counting function of \( f \).

**THEOREM B.** Let \( D \) be compact Riemann surface with smooth boundary \( \partial D \), \( f : D \rightarrow M \) is a holomorphic mapping, then

\[ \chi(D) - \frac{1}{2\pi} \int_{\partial D} K + n_1(D) = \chi(M)v(D), \]  \hspace{1cm} (3)

where \( \chi(D) \) and \( \chi(M) \) are Euler's characteristics of \( D \) and \( M \) respectively, \( n_1(D) \) is the stationary index of \( f \) in \( D \).

For the case of holomorphic mappings, S.S. Chern gave the integral form of theorem A and theorem B, and also proved the relation inequality of deficient values. In this paper, we replace \( f \) by differential mapping, and also get similar results. We have the following main result.

**THEOREM 1.** Let \( D \) be compact Riemann surface with smooth boundary \( \partial D \), if \( f : D \rightarrow M \) is a differential mapping, and if the critical points of \( f \) are all isolated points, \( f \) is orientation-preserving except critical points. Then we also have equality (3).

2. The Proof of Theorem 1.

By using local coordinate \( z = x + yi \), we have \( G = gdzd\bar{z} = g(dx^2 + dy^2) \), and \( \Omega = \frac{i}{2} gdz \wedge d\bar{z} = \frac{1}{2} gdx \wedge dy \), where \( g \) is a positive function which belongs to \( C^\infty \).

**Lemma 1 (Gauss-Bonnet formula).** If \( A \) is a compact subset of \( M \) with smooth boundary \( \partial A \), let \( K = Kgds \) be the curvature form of \( \partial A \) about \( G \), where \( Kg \) is the curvature of \( \partial A \), then

\[ 2\pi \chi(A) - \int_{\partial A} K = \int_{A} K \Omega, \]

especially, \( \chi(M) = \frac{1}{2\pi} \int_{M} K \Omega. \)

Now we define stationary index \( n_1(D) \) of differential mapping of \( f : D \rightarrow M \) as the following:

We suppose that \( \alpha_1, \alpha_2, \ldots, \alpha_n \) are the critical points of \( f \) in \( D - \partial D \) (i.e. \( df(\alpha_j) = 0, j = 1,2,\ldots,n \)). Because \( f \) is orientation-perserving except the critical points, then the metric of \( G \) on \( M \) can induce Hermitian metric \( f^*G \) on \( D - \{ \alpha_1, \ldots, \alpha_n \} \), \( f : D - \{ \alpha_1, \ldots, \alpha_n \} \rightarrow M \) is local isometry mapping, so \( f^*(K) \) is equal to the product of Gauss curvature of \( f^*G \) on \( D - \{ \alpha_1, \ldots, \alpha_n \} \) and volume element of \( f^*G \). We suppose that \( z_i \) is the local coordinate function in the neighborhood of \( \alpha_i \) such that \( z_i(\alpha_i) = 0, W_j = \{|z| < \epsilon\}, W = \bigcup_{j=1}^{n} W_j \). We use \( K \) denote the geodesic curvature form of \( \partial W \) about \( f^*G \).

Now we define stationary index \( I_{a_i} \) of \( f \) at \( \alpha_i \) as the following:

\[ I_{a_i} = \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_{\partial W_i} K - 1, \]

and call \( n_1(D) = \sum_{j=1}^{n} I_{a_j} \) is the stationary index of \( f \) in \( D \). We apply the lemma 1 to the metric of \( f^*G \) in \( D - W \), then

\[ 2\pi \chi(D - M) - \int_{\partial D} K + \int_{\partial W} K = \int_{D - M} f^*(K), \]

where the orientation of \( \partial W \) is induced by the orientation of \( W \). Clearly, \( \chi(D - M) = \chi(D) - n \), and \( \lim_{\epsilon \to 0} \int_{D - W} f^*(K) = \int_{D} f^*(K) \). So

\[ \int_{D} f^*(K) = 2\pi \chi(D) - \int_{\partial D} K + 2\pi \lim_{\epsilon \to 0} \sum_{j=1}^{n} \left( \frac{1}{2\pi} \int_{\partial W_j} K - 1 \right) \]

\[ = 2\pi \chi(D) - \int_{\partial D} K + 2\pi n_1(D). \]

We notice that \( K \) is constant, then apply Gauss-Bonnet formula, we have:
So we have

\[ \frac{1}{2\pi} \int_D f^*(K\Omega) = \frac{KA}{2\pi} \int_D f^*\Omega = \chi(M)v(D). \]

We are done.

3. Integral Form of Theorem 1.

Let \( V \) be a open Riemann surface. Suppose that \( V \) has an infinite harmonic exhaustion function \( \tau \) [13]. We also suppose that \( f : V \mapsto M \) is a differential mapping, and all critical points are isolated points, \( f \) is orientation-preserving (except critical points), if \( \varphi \) is one-form on \( V \), let \( \ast \varphi \) be the conjugate one-form.

We let \( V[r] = \{ p | \tau(p) \leq r \} \), if \( r \) is not the critical values of \( \tau \), then \( V[r] \) is compact subset in \( V \) with smooth boundary. Let \( n(r,a) = n(V[r], a) \), \( v(r) = v(V[r]) \), \( \chi(r) = \chi(V[r]) \), \( n_1(r) = n_1(V[r]) \).

For \( f \), we use theorem 1, we conclude

\[ \chi(r) - \frac{1}{2\pi} \int_{\partial V[r]} K + n_1(r) = \chi(M)v(r), \]

where \( K \) is geodesic curvature form of \( \partial V[r] \) about \( f^*\Gamma \). We can introduce function \( h \) such that, \( f^*\Omega = h dr \wedge \ast dr \) on \( V - V[r(\tau)] \), because \( f \) is orientation-persveing, so \( f^*\Omega \) and \( V \) have the same orientation, clearly, \( dr \wedge \ast dr \) and \( V \) have the same orientation, so \( h \) is nonnegative function, and \( K = \frac{1}{2} d^2 \log h \), then we have \( \int_{\partial V[r]} K = \frac{1}{2} \int_{\partial V[r]} d^2 \log(h) \). According to [13], we can use special coordinate function \( \sigma = \tau + i\rho \), so

\[ d^2 \log(h) = -\frac{\partial \log(h)}{\partial \sigma} \, d\sigma + \frac{\partial \log(h)}{\partial \tau} \, d\tau. \]

and

\[ \frac{1}{2} \int_{\partial V[r]} d^2 \log(h) = \frac{1}{2} \int_{\partial V[r]} \frac{\partial \log(h)}{\partial \tau} \, d\rho = \frac{\partial}{\partial \tau} \left( \frac{1}{2} \int_{\partial V[r]} \log(h) \ast dr \right). \]

By using the method which we deal with holomorphic functions, we can introduce the following functions: \( E(r) = \int_{r_0}^r x(t) \, dt \), \( N_1(r) = \int_{r_0}^r n_1(t) \, dt \), and \( T(r) = \int_{r_0}^r v(t) \, dt \), where \( r > r_0 \geq r(\tau) \).

Because of (5), we have

\[ E(r) + N_1(r) - \frac{1}{4\pi} \int_{\partial V[r]} (\log(h)) \ast dr |_{r_0}^r = \chi(M)T(r). \]

This is the integral form of theorem 1.

References


