COMPLETELY POSITIVE LINEAR OPERATORS FOR BANACH SPACES

MINGZE YANG

Department of Mathematics
University of Saskatchewan
Saskatoon, Sask. Canada S7N 0W0

(Received January 9, 1992 and in revised form February 25, 1992)

ABSTRACT. Using ideas of Pisier, the concept of complete positivity is generalized in a different direction in this paper, where the Hilbert space $H$ is replaced with a Banach space and its conjugate linear dual. The extreme point results of Arveson are reformulated in this more general setting.

KEY WORDS AND PHRASES: Banach spaces, completely positive operators, extreme points, pure elements.

1980 AMS SUBJECT CLASSIFICATION CODES: 46L05, 47A67

1. INTRODUCTION.

In [6], Pisier studied completely bounded maps from a $C^*$-algebra to $B(X,Y)$, the space of bounded operators between two arbitrary Banach spaces $X$ and $Y$. Of course, there is a generalization of ordinary completely bounded maps. In this paper, we first define complete positivity for a map from $C^*$-algebra to $B(X,X^*)$, where $X^*$ denotes the antilinear dual space of $X$ (the set of all conjugate linear functionals on $X$). Then we give a representation theorem, and give complete solutions to three extremal problems.

In this paper, the $C^*$-algebra $A$ always has an identity.

2. COMPLETELY POSITIVE OPERATORS.

DEFINITION 2.1. Let $X$ be a Banach space, and $T \in B(X,X^*)$. We call $T$ positive if, for all positive integers $n$ and $x_1, \ldots, x_n \in X$, we have

$$
\sum_{i=1}^{n} \sum_{j=1}^{n} T(x_i)(x_j) \geq 0.
$$

REMARK 2.2. We have $T'(X)^* = T(X')$, and so $M_n(B(X,X^*)) = B(T_n'(X),T_n'(X^*)) = B(T_n(X),T_n(X^*))$. Thus we may define positivity for $M_n(B(X,X^*))$.

DEFINITION 2.3. Let $A$ be a $C^*$-algebra, $\phi$ a linear map from $A$ to $B(X,X^*)$ and let $\phi_n(a_n) = (\phi(a_n))$ for $(a_n) \in M_n(A)$. If $\phi_n$ is positive for all $n$, then we say $\phi$ is completely positive.

THEOREM 2.4. Let $\phi : A \rightarrow B(X,X^*)$ be a completely positive map. There is a Hilbert space $\mathcal{H}$, a representation $\pi$ of $A$ on $\mathcal{H}$ and a bounded operator $V \in B(X,\mathcal{H})$ such that, for all $a \in A$,

$$
\phi(a) = V^* \pi(a)V,
$$

and $\mathcal{H} = [\pi(A)VX]$, where $V^*(h)(x) = < h, V(x) >$, for all $h \in \mathcal{H}, x \in X$. 

PROOF: Consider the vector space tensor product \( A \otimes X \) and define a bilinear form as follows:

If \( u = x_1 \otimes \xi_1 + \ldots + x_m \otimes \xi_m, \ v = y_1 \otimes \eta_1 + \ldots + y_n \otimes \eta_n, \)

\[
< u, v > = \sum_{i,j} (\phi(y_i^*x_j)(\xi_j))(\eta_i).
\]

Because \( \phi \) is completely positive, we have the fact that \( <,> \) is positive semi-definite. For each \( a \in A \), define a linear transformation \( \pi_0(a) \) on \( A \otimes X \) by

\[
\pi_0(a)(\sum^n_{j=1} x_j \otimes \xi_j) = \sum^n_{j=1} (ax_j) \otimes \xi_j.
\]

\( \pi_0 \) is an algebra homomorphism for which

\[
< u, \pi_0(a)v > = < \pi_0(a^*)u, v >
\]

for all \( u, v \in A \otimes X \).

For fixed \( u \), \( \rho(a) = < \pi_0(a^*)u, u > \) defines a positive linear functional on \( A \); i.e., \( \rho(a^*a) \geq 0 \). Hence, \( < \pi_0(a)u, \pi_0(a)u > = < \pi_0(a^*a)u, u > = \rho(a^*a) \leq \|a^*a\|\rho(1) = \|a\|^2 < u, u > \), where 1 is the identity of \( A \).

Now let \( R = \{ u \in A \otimes X : < u, u > = 0 \} \). \( R \) is a linear subspace \( A \otimes X \), invariant under \( \pi_0(a) \), for all \( a \in A \). So \( <,> \) determines a positive definite inner product on the quotient \( (A \otimes X)/R \) in the usual way.

Let \( \mathcal{H} = (A \otimes X)/R \). There is a unique representation \( \pi \) of \( A \) on \( \mathcal{H} \) such that

\[
\pi(a)(u + R) = \pi_0(a)u + R
\]

\( a \in A, u \in A \otimes X \).

We define a linear map \( V: X \rightarrow \mathcal{H} \) by

\[
V(\xi) = 1 \otimes \xi + R
\]

for all \( \xi \in X \).

We may verify that \( V \) is bounded, and \( \phi(a) = \overline{V^*\pi(a)V} \) for all \( a \in A \).

Let \( R_1 = [\pi(A)VX] \subseteq \mathcal{H} \), and \( \pi_1(a) = \pi(a)|_{R_1} \) for all \( a \in A \). Because \( \pi(1) = I \), so \( V(X) \subseteq R_1 \).

We have \( \overline{V^*\pi(a)V(x_1)} = \overline{V^*\pi(a)|_{R_1}V(x_1)} = \overline{V^*\pi_1(a)V(x_1)} = \phi(a)(x_1) \), for all \( x_1 \in X, a \in A \). So we may assume that \( \mathcal{H} = [\pi(A)VX] \).

Suppose \( \phi: A \rightarrow B(X, X^*) \) is a completely positive map. If there exists Hilbert spaces \( \mathcal{H}_i \), representations \( \pi_i \) of \( A \) on \( \mathcal{H}_i \), and bounded operators \( V_i : X \rightarrow \mathcal{H}_i \) then

\[
\phi(a) = \overline{V_i^*\pi_i(a)V_i},
\]

for \( i = 1, 2 \), where \( \mathcal{H}_i = [\pi_i(A)V_iX] \). Define \( U: \mathcal{H}_1 \rightarrow \mathcal{H}_2 \) by

\[
U(\sum^n_{i=1} a_i V_1 \xi_i) = \sum^n_{i=1} a_i V_2 \xi_i,
\]

for all \( a_1, \ldots, a_n \in A, \xi_1, \ldots, \xi_n \in X \). Then we need to extend to \( \mathcal{H}_1 \). We may verify that \( UV_1 = V_2 \) and \( U \pi_1(a) = \pi_2(a)U \) for all \( a \in A \).
Next we verify that $U$ is an unitary.

\[
\begin{align*}
\langle \sum_{i=1}^{n} \pi_2(a_i)V_2\xi_i, \sum_{i=1}^{n} \pi_2(a_i)V_2\xi_i \rangle &= \sum_{i,j} \langle \pi_2(a_i)V_2\xi_i, \pi_2(a_j)V_2\xi_j \rangle \\
&= \sum_{i,j} \langle \pi_2(a_i^*a_i)V_2\xi_i, V_2\xi_j \rangle \\
&= \sum_{i,j} \phi(a_i^*a_i)\langle \xi_i, \xi_j \rangle \\
&= \sum_{i,j} \langle \pi_1(a_i^*a_i)V_1\xi_i, V_1\xi_j \rangle \\
&= \langle \sum_{i} \pi_1(a_i)V_1\xi_i, \sum_{i} \pi_1(a_i)V_1\xi_i \rangle.
\end{align*}
\]

So the representation given in Theorem 2.4 is unique up to unitary equivalence.

3. PREPARATIONS.

**NOTATION 3.1.** Let $CP(A, X)$ denote all completely positive linear maps from $A$ to $B(X, X)$.  

**LEMMA 3.2.** Let $\phi_1$ and $\phi_2$ belong to $CP(A, X)$, and suppose that $\phi_1 \leq \phi_2$. Let $\phi(a) = \sum_{i} \pi_i(a) V_i$ be the canonical expression of $\phi$, where $\pi_i$ is a representation of $A$ on $R_i$ such that $[\pi_i(A) V_i X] = R_i$, $i = 1, 2$. Then there exists a contraction $T \in B(R_2, R_1)$ such that

\[
TV_2 = V_1,
\]

\[
T\pi_2(a) = \pi_1(a)T
\]

for all $a \in A$.

**PROOF:** For every $f_1, ..., f_n, x_1, ..., x_n \in A$,

\[
\| \sum_{i=1}^{n} \pi_1(a_i) V_i \xi_i \|^2 = \langle \sum_{i=1}^{n} \pi_1(a_i) V_i \xi_i, \sum_{i=1}^{n} \pi_1(a_i) V_i \xi_i \rangle \\
= \sum_{i,j} \langle \pi_1(a_i^*a_i) V_i \xi_i, \xi_j \rangle \\
= \sum_{i,j} \langle \phi_i(a_i^*a_i) \xi_i, \xi_j \rangle \\
\leq \sum_{i,j} \langle \phi_2(a_i^*a_i) \xi_i, \xi_j \rangle \\
= \| \sum_{i=1}^{n} \pi_2(a_i) V_2 \xi_i \|^2
\]

Define $T: R_2 \rightarrow R_1$ by

\[
T(\sum_{i=1}^{n} \pi_2(a_i) V_2 \xi_i) = \sum_{i=1}^{n} \pi_1(a_i) V_1 \xi_i
\]

We can verify that above two statements hold.

**NOTATION 3.3.** For $\phi \in CP(A, X)$, let $[0, \phi] = \{ \psi \in CP(A, X); \psi \leq \phi \}$. Let $\phi(a) = \sum_{i} \pi_i(a) V_i$ for all $a \in A$. For each operator $T \in \pi(A)^t$, define a map $\phi_T(a) = \sum_{i} \pi_i(a) \xi_i V_i$. Then $T \rightarrow \phi_T$ is linear. If $\phi_T = 0$, we have

\[
\langle T\pi(a)V_\xi, \pi(b)V_\eta \rangle = \langle T\pi(b^*a)V_\xi, V_\eta \rangle = \phi_T(b^*a)(\xi)(\eta) = 0
\]

\[
\langle T(\sum_{i=1}^{n} \pi_i(a_i) V_\xi_i) \sum_{i=1}^{n} \pi(b_j) V_\xi_j \rangle = 0.
\]

So $T = 0$. That is, $T \rightarrow \phi_T$ is injective.
THEOREM 3.4. $T \rightarrow \phi_T$ is an affine order isomorphism of the partially ordered convex set of $\{T \in \pi(A)': 0 \leq T \leq I\}$ onto $[0, \phi]$.

The proof of this theorem is exactly the same way as the proof of theorem in Arveson’s paper [1].

4. THE THREE EXTREMAL PROBLEMS.

Now we come to discuss three extremal problems.

DEFINITION 4.1. A completely positive map $\phi \in CP(A, X)$ is pure if, for every $\psi \in CP(A, X)$, $\psi \leq \phi$ implies that $\psi$ is a scalar multiple of $\phi$.

REMARK 4.2. According to [3], the extreme rays of $CP(A, X)$ can be characterized as the half lines $\{t\phi: t \geq 0\}$, where $\phi$ is a pure element of $CP(A, X)$.

We state the following theorems without proofs, for the proofs are almost the same as those in Arveson’s paper [1].

THEOREM 4.3. All nonzero pure elements of $CP(A, X)$ are precisely those of the form $\phi(a) = V^*\pi(a)V$, where $\pi$ is an irreducible representation of $A$ on some Hilbert space $R$ and $V \in B(X, R)$, such that $R = [\pi(A)VX]$.

THEOREM 4.4. Let $\phi \in CP(A, X)$ and let $\phi(a) = V^*\pi(a)V$ be its canonical representation. The extreme points of $[0, \phi]$ are those maps of the form $V^*P\pi(a)V$, where $P$ is a projection in $\pi(A)'$.

We consider the extreme points of the set $CP(A, X; K) = \{\phi \in CP(A, X); \phi(1) = K\}$, where $K$ is a fixed positive operator in $B(X, X^*)$.

THEOREM 4.5. Let $\phi \in CP(A, X; K)$ and let $\pi(a) = V^*\pi(a)V$ be its canonical representation with $V^*V = K$. Then $\phi$ is an extreme point of $CP(A, X; K)$ if and only if $[VX]$ is a faithful subspace for the commutant $\pi(A)'$ of $\pi(A)$.

ACKNOWLEDGEMENT. The author thanks professor K. F. Taylor for helpful suggestions.

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