FLOWS FOR CHOSEN VORTICITY FUNCTIONS—
EXACT SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

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ABSTRACT. Solutions are obtained for the equations of the motion of the steady incompressible viscous planar generalized Beltrami flows when the vorticity distribution is given by $\nabla^2 \psi = \psi + f(z, y)$ for three chosen forms of $f(z, y)$.

KEY WORDS AND PHRASES. viscous flow, asymptotic suction profile, Beltrami flow.


1. INTRODUCTION.

Only a small number of exact solutions of the Navier-Stokes equations has been found and Chang-Yi Wang [1] has given an excellent review of these solutions. These known solutions of viscous incompressible Newtonian fluids may be classified into three types:

(i) Flows for which the non-linear inertia terms in the linear momentum equations vanish identically. Parallel flows and flows with uniform suction are examples of these flows;

(ii) flows with similarity properties such that the flow equations reduce to a set of ordinary differential equations. Stagnation point flow is an example of such flows;

(iii) flows for which the vorticity function is so chosen that the governing equation in terms of the stream function reduces to a linear equation. Taylor [2], Kampe de Feriet [3], Kovasznay [4], Wang [5] and Lin and Tobak [6] employed this approach, taking $\nabla^2 \psi = K \psi$, $\nabla^2 \psi = f(\psi)$. $\nabla^2 \psi = y + (K^2 - 4\pi^2)\psi$, $\nabla^2 \psi = A \psi + C \psi$ and $\nabla^2 \psi = K(\psi - Ry)$, respectively.

In this paper, we study generalized Beltrami flows when the vorticity function $\omega = -\nabla^2 \psi$ is given by $\nabla^2 \psi = \psi + Ay^2 + Bz + Cz + Dy$, $\nabla^2 \psi = \psi + Ay^2 + Cz + Dz$, $\nabla^2 \psi = \psi + Cz + Dy$, where $A, B, C, D$ are real constants.
2. BASIC EQUATIONS AND SOLUTIONS.

Steady plane incompressible viscous fluid flow, in the absence of external forces, is governed by the system:

\[
\begin{align*}
\ddot{u} + \dot{v} &= 0 \\
\dddot{u} + \ddot{v} + \frac{1}{\rho} \dddot{p} &= \mu \nabla^2 \dddot{u} \\
\dddot{v} + \ddot{u} + \frac{1}{\rho} \dddot{p} &= \mu \nabla^2 \dddot{v}
\end{align*}
\]

(2.1)

where \(\dddot{u}(\dddot{x}, \dddot{y})\), \(\dddot{v}(\dddot{x}, \dddot{y})\) are the velocity components, \(\dddot{p}(\dddot{x}, \dddot{y})\) the pressure function, \(\rho\) the constant density, \(\mu\) the constant viscosity and \(\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\) is the Laplacian operator. The vorticity function for this flow is given by

\[
\omega = \dddot{u} - \dddot{v}
\]

(2.2)

Letting \(U, L\) to be the characteristic velocity and length respectively, we introduce the non-dimensional variables

\[
x = \frac{\dddot{x}}{L}, \quad y = \frac{\dddot{y}}{L}, \quad u = \frac{\dddot{u}}{U}, \quad v = \frac{\dddot{v}}{U}, \quad \omega = \frac{L\dddot{\omega}}{U}, \quad p = \frac{\dddot{p}}{\rho U^2}
\]

(2.3)

in system (2.1) and equation (2.2). We apply the integrability condition \(p_{xy} = p_{yx}\) to the linear momentum equations to find that \(u, v, \omega\) must satisfy the system:

\[
\begin{align*}
u_x + v_y &= 0 \\
u \omega_x + v \omega_y &= \frac{1}{R} \nabla^2 \omega \\
u_z - u_y &= \omega
\end{align*}
\]

(2.4)

where \(R = \frac{UL}{\rho}\) is the Reynolds number.

Introducing the stream function \(\psi(x, y)\) such that

\[
u = \psi_y, \quad v = -\psi_x
\]

(2.5)

in system (2.4), we find that \(\psi(x, y)\) must satisfy

\[
\nabla^4 \psi + R \frac{\partial(\psi, \nabla^2 \psi)}{\partial(x, y)} = 0
\]

(2.6)

In this paper, we study flows for which the vorticity distributions take the forms

\[
\begin{align*}
(a) \quad \omega &= -\nabla^2 \psi = -(\psi + Ay^2 + Bxy + Cz + Dy) \\
(b) \quad \omega &= -\nabla^2 \psi = -(\psi + Ay^2 + Cz + Dy) \\
(c) \quad \omega &= -\nabla^2 \psi = -(\psi + Cz + Dy)
\end{align*}
\]

(2.7) (2.8) (2.9)

where \(A, B, C, D\) are real constants.

Form (a):

Substituting (2.7) in the compatibility equation (2.6), we get

\[
R(2Ay + Bx + D)\psi_x - R(By + C)\psi_y + \psi + Ay^2 + Bxy + Cz + Dy + 2A = 0
\]

(2.10)
Employing the canonical coordinates
\[
\zeta = Ay^2 + Bxy + Cz + Dy, \quad \eta = y
\] (2.11)
where \((By + C) \neq 0\), (2.10) may be written as
\[
-R(B\eta + C)\psi + \psi + \zeta + 2A = 0.
\] (2.12)

This equation is solved to obtain
\[
\psi = f(\zeta)(By + D)t - (Ay^2 + Bxy + Cz + Dy + 2A)
\] (2.13)
where \(f\) is an arbitrary function of \(\zeta\). Introducing (2.13) into (2.7), we get
\[
\{R^2 \left[ C^2(C^2 + D^2) + 2BCD\xi + B^2\xi^2 \right] f''(\xi) + 2R[C(RAC + D) - B\xi] f'(\xi)
+ [1 - RB - R^2C^2] f(\xi) \} + 2RC \left[ 2R[C(AD + BC) + AB\xi] f''(\xi)
+ 2A[R^2 + 1] f'(\xi) - RBf(\xi) \right] \eta^2
\]
\[
+ 2A[R^2 + 1] f'/(\xi) - RBf(\xi) \right] \eta^2
\]
\[
+ 4R^2BC \left\{ [A^2 + B^2] f''(\xi) \right\} \eta^4 = 0
\] (2.14)

Since \(\zeta, \eta\) are independent variables and \(\{1, \zeta, \xi, \eta^2, \eta^3, \eta^4\}\) is a linearly independent set, it follows that the coefficients of the various powers of \(\eta\) are zero. Taking the coefficients of \(\eta^4, \eta^3, \eta^2, \eta\) and 1 equal to zero, we get
\[
f(\zeta) = c_1\zeta + c_2
\] (2.15)
\[
2A(RB + 1)c_1 - RBc_2 - RBc_1\zeta = 0
\] (2.16)
where \(c_1, c_2\) are arbitrary constants. Since \(\{1, \zeta\}\) is a linearly independent set, it follows from (2.16) that \(2A(RB + 1)c_1 - RBc_2 = 0, RBc_1 = 0\) giving \(c_1 = c_2 = 0\). Using \(c_1 = c_2 = 0\) in (2.15), we obtain \(f(\zeta) = 0\).

From (10), the stream function is given by
\[
\psi(x, y) = -(Ay^2 + Bxy + Cz + Dy + 2A)
\] (2.17)
The exact integral of this flow is
\[
u = -(2Ay + Bx + D), \quad v = By + C, \quad \text{and}
\]
\[
p = p_0 - \frac{1}{2} \left[ B^2(x^2 + y^2) + 2(BD - 2AC)x + 2BCy \right]
\] (2.18)
where \(p_0\) is an arbitrary constant.

Equation (2.17) represents an impingement of two constant-vorticity oblique flows with stagnation point
\[
(x, y) = \left( \frac{2AC - BD}{B^2}, -\frac{C}{B} \right)
\] (2.19)
for non-zero values of $A, B, C$ and $E$. The stagnation point shifts upward as $B$ gets smaller for fixed values of $A, C$ and $E$. We remark that when $A = B = -1, C = D = 0$, the solution (2.17) reduces to one of the flows in Wang's [1] paper.

Form (b):

Employing (2.8) in (2.6), we obtain

$$R(2Ay + D)\psi_x - RC\psi_y + \psi + Ay^2 + Cz + Dy + 2A = 0 \quad (2.20)$$

Choosing the canonical coordinates

$$\xi = Ay^2 + Cz + Dy, \quad \eta = y \quad (2.21)$$

where $C \neq 0$, (16) takes the form

$$-RC\psi_{\eta} + \psi + \xi + 2A = 0. \quad (2.22)$$

We solve this equation to get

$$\psi = g(\xi) \exp \left( \frac{1}{RC} \right) - (Ay^2 + Cz + Dy + 2A) \quad (2.23)$$

where $g$ is an arbitrary function of $\xi$. We substitute (2.23) into (2.8) to get

$$\left[ R^2 C^4 g''(\xi) + 2R^2 AC^2 g'(\xi) + (1 - R^2 C^2 g(\xi)) + 2RCg'(\xi)(2A\eta + D) 
+ R^2 C^2 g''(\xi)(2A\eta + D)^2 \right] = 0 \quad (2.24)$$

Since $\xi, \eta$ are independent variables and $\{1, (2A\eta + D), (2A\eta + D)^2\}$ is a linearly independent set, it follows that

$$g''(\xi) = 0, \quad g'(\xi) = 0, \quad (1 - R^2 C^2)g(\xi) = 0 \quad (2.25)$$

From $(1 - R^2 C^2)g(\xi) = 0$, we get the three possibilities: $g(\xi) = 0, R^2 C^2 \neq 1; R^2 C^2 = 1, g(\xi) \neq 0; g(\xi) = 0, R^2 C^2 = 1$.

The stream function (2.23) is given by

$$\psi(x, y) = \begin{cases} 
-(Ay^2 + Cz + Dy + 2A) & ; g = 0, \quad R^2 C^2 \neq 1 \\
K \exp \left( \frac{1}{RC} \right) - (Ay^2 + Cz + Dy + 2A); \quad R^2 C^2 = 1, \quad g \neq 0 \\
-(Ay^2 + Cz + Dy + 2A) & ; g = 0, \quad R^2 C^2 = 1 
\end{cases} \quad (2.26)$$

where $g \neq 0$ implies $g = K$ (non-zero constant).

When the stream function is given by

$$\psi(x, y) = -(Ay^2 + Cz + Dy + 2A); \quad R^2 C^2 = 1 \quad \text{or} \quad R^2 C^2 \neq 1, \quad (2.27)$$

the exact integral for the flow is

$$u = -(2Ay + D), \quad v = C, \quad \text{and} \quad p = p_0 + 2ACz \quad (2.28)$$
where $p_0$ is an arbitrary constant.

The solution (2.28) may be realized on a plate situated along $y = -\frac{D}{2A}$ with uniform suction or blowing. $C > 0$ and $C < 0$, respectively, for blowing and suction at the plate.

The exact integral for the flow given by the stream function

$$\psi(x, y) = K \exp \left( \frac{1}{RC} y \right) - (Ay^2 + Cx + Dy + 2A); \quad R^2C^2 = 1$$

is

$$u = \frac{K}{RC} \exp \left( \frac{1}{RC} y \right) - (2Ay + D), \quad v = C, \quad \text{and} \quad p = p_0 + 2ACz$$

where $p_0$ is an arbitrary constant.

If $K = RCD$ in (2.29) and (2.30), the velocity profile in (2.30) can be realized on a plate located along $y = 0$ with uniform suction. The velocity profile attains the form

$$u = D \exp \left( \frac{1}{RC} y \right) - (2Ay + D), \quad v = C$$

only asymptotically, and so may be regarded as the asymptotic suction profile [7]. $C > 0$ and $C < 0$ for blowing and suction at the plate, respectively.

Form (c):

Substitution of (2.8) into (2.6) yields

$$RD\psi_x - RC\psi_y + \psi + Cx + Dy = 0$$

The canonical coordinates

$$\xi = Cx + Dy, \quad \eta = y; \quad C \neq 0$$

are employed in (2.32) to get

$$-RC\psi_x + \psi + \xi = 0.$$

The solution of this equation is

$$\psi = h(\xi) \exp \left( \frac{1}{RC} y \right) - (Dx + Ey)$$

where $h$ is an arbitrary function of $\xi$. We employ (2.34) in (2.9) to obtain

$$R^2C^2(C^2 + D^2)h''(\xi) + 2RCDh'(\xi) + (1 - R^2C^2)h(\xi) = 0$$

The general solution of (2.35) is

$$h(\xi) = \begin{cases} A_1 \exp(\lambda_1 \xi) + A_2 \exp(\lambda_2 \xi) & ; R^2(C^2 + D^2) - 1 > 0 \\ (B_1 + B_2 \xi) \exp \left( -\frac{RD}{C} \xi \right) & ; R^2(C^2 + D^2) - 1 = 0 \\ C_1 \cos(m \xi + C_2) \exp \left[ -\frac{D}{RC(C^2 + D^2)} \xi \right] & ; R^2(C^2 + D^2) - 1 < 0 \end{cases}$$

(2.36)
where
\[
\lambda_{1,2} = \frac{-D \pm C \sqrt{R^2(C^2 + D^2) - 1}}{RC(C^2 + D^2)}, \quad m = \frac{\sqrt{1 - R^2(C^2 + D^2)}}{R(C^2 + D^2)}
\]
(2.37)
and \(A_1, A_2, B_1, B_2, C_1, C_2\) are arbitrary constants.

We shall study these three possibilities separately.

(i) \(R^2(C^2 + D^2) - 1 > 0\)

The stream function, from (2.34) and (2.36), is
\[
\psi(z, y) = A_1 \exp \left[ \lambda_1 Cz + \left( \lambda_1 D + \frac{1}{RC} \right) y \right] + A_2 \exp \left[ \lambda_2 Cz + \left( \lambda_2 D + \frac{1}{RC} \right) y \right] - (Cz + Dy)
\]
(2.38)
The exact integral of this flow is
\[
\begin{align*}
  u &= \left( \lambda_1 D + \frac{1}{RC} \right) A_1 \exp \left[ \lambda_1 Cz + \left( \lambda_1 D + \frac{1}{RC} \right) y \right] \\
  &+ \left( \lambda_2 D + \frac{1}{RC} \right) A_2 \exp \left[ \lambda_2 Cz + \left( \lambda_2 D + \frac{1}{RC} \right) y \right] - D, \\
  v &= -D \left\{ \lambda_1 A_1 \exp \left[ \lambda_1 Cz + \left( \lambda_1 D + \frac{1}{RC} \right) y \right] \\
  &+ \lambda_2 A_2 \exp \left[ \lambda_2 Cz + \left( \lambda_2 D + \frac{1}{RC} \right) y \right] - 1 \right\},
\end{align*}
\]
(2.39)
and
\[
\begin{align*}
  p &= p_0 + 2 \left[ 1 - \frac{1}{R^2(C^2 + D^2)} \right] A_1 A_2 \exp \left[ \frac{2(Dy - Ex)}{R^2(C^2 + D^2)} \right]
\end{align*}
\]
where \(p_0\) is an arbitrary constant and \(\lambda_1, \lambda_2\) are given by (2.37).

This flow represents an impingement of an oblique uniform stream with an oblique rotational, divergent flow, with stagnation point
\[
(z, y) = \frac{RC}{2\sqrt{R^2(C^2 + D^2) - 1}} \left( C \ln \left( \frac{-A_1}{A_2} \right) \\
- D \sqrt{R^2(C^2 + D^2) - 1} \ln \left\{ \frac{-4A_1 A_2 [R^2(C^2 + D^2) - 1]}{R^2(C^2 + D^2)^2} \right\} \\
D \ln \left( \frac{-A_1}{A_2} \right) + C \sqrt{R^2(C^2 + D^2) - 1} \ln \left\{ \frac{-4A_1 A_2 [R^2(C^2 + D^2) - 1]}{R^2(C^2 + D^2)^2} \right\} \right)
\]
(2.40)
where \(A_1, A_2\) are non-zero real constants and either \(A_1 > 0, A_2 < 0\) or \(A_1 < 0, A_2 > 0\). For fixed values of \(R, C\) and \(D\), the stagnation point shifts upward when the absolute value of \(A_2\) is larger than that of \(A_1\).

If \(A_1\) and \(A_2\) are of the same sign, the above phenomenon does not take place, and we have a flow without a stagnation point.

(ii) \(R^2(C^2 + D^2) - 1 = 0\)

Using (2.36) in (2.34), the stream function is
\[
\psi(z, y) = [B_1 + B_2(Cz + Dy)] \exp[R(Cy - Dz)] - (Cz + Dy)
\]
(2.41)
This flow has the exact integral
\[
\begin{align*}
  u &= \{ DB_2 + RC [B_1 + B_2(Cz + Dy)] \} \exp[R(Cy - Dz)] - E, \\
  v &= \{- DB_2 + RD [B_1 + B_2(Cz + Dy)] \} \exp[R(Cy - Dz)] + D, \quad \text{and} \\
  p &= p_0 - \frac{1}{2R^2} B_1^2 \exp[2R(Cy - Dz)]
\end{align*}
\]
(2.42)
where $p_0$ is an arbitrary constant.

If $B_2$ is a positive real constant, this flow represents an impingement of an oblique uniform stream with an oblique rotational, divergent flow, with stagnation point

$$ (x, y) = \left( \frac{1}{C^2 + D^2} \left( \frac{CB_1}{B_2} - \frac{D}{R} \ln B_2, \frac{DB_1}{B_2} + \frac{C}{R} \ln B_2 \right) \right) \quad (2.43) $$

For fixed values of $R$ and $C$, the stagnation point shifts upward if $B_1$ and $D$ are of opposite signs and the absolute value of $B_1$ is larger than $B_2$.

If $B_2$ is a negative real constant, (2.41) represents an oblique uniform stream which abuts on an oblique rotational, convergent flow.

(iii) $R^2(C^2 + D^2) - 1 < 0$

From (2.27) and (2.36), the stream function is given by

$$ \psi(x, y) = C_1 \cos \left[ m(Cx + Dy) + C_2 \right] \exp \left[ \frac{Cy - Dz}{R(C^2 + D^2)} \right] - (Cx + Dy) \quad (2.44) $$

The exact integral for this flow is

$$ u = \frac{C_1}{R(C^2 + D^2)} \{ C \cos[m(Cx + Dy) + C_2] 
- mRD(C^2 + D^2) \sin[m(Cx + Dy) + C_2] \} \exp \left[ \frac{Cy - Dz}{R(C^2 + D^2)} \right] - D, $$

$$ v = \frac{C_1}{R(C^2 + D^2)} \{ D \cos[m(Cx + Dy) + C_2] 
+ mRC(C^2 + D^2) \sin[m(Cx + Dy) + C_2] \} \exp \left[ \frac{Cy - Dz}{R(C^2 + D^2)} \right] + C, \quad \text{and} $$

$$ p_0 = \frac{1}{2} \left[ 1 - \frac{1}{R^2(C^2 + D^2)} \right] C_1^2 \cos 2[m(Cx + Dy) + C_2] \exp \left[ \frac{2(Cy - Dz)}{R(C^2 + D^2)} \right] $$

where $p_0$ is an arbitrary constant, and $m$ is given by (2.37).

If $C_1 > 0$, the stagnation points for this flow are

$$ (x, y) = \left( \frac{RC[(2n + 1)\frac{\pi}{2} - C_2]}{\sqrt{1 - R^2(C^2 + D^2)}}, \frac{RD[(2n + 1)\frac{\pi}{2} - C_2]}{\sqrt{1 - R^2(C^2 + D^2)}} \right) - RD \ln \left[ \frac{C_1 \sqrt{1 - R^2(C^2 + D^2)}}{R(C^2 + D^2)} \right] $$

where $n$ is an integer.

Fig. 1 shows the streamlines for $\psi(x, y) = -(Ay^2 + Bx + Cz + Dy + 2A)$ when $A = B = C = D = 1$. Figures 2 and 3 represent the flows $\psi(x, y) = -(Ay^2 + Cx + Dy + 2A)$ and $\psi(x, y) = K \exp \left( \frac{\partial}{\partial y} \right) - (Ay^2 + Cx + Dy + 2A)$ for $K = R = A = C = D = 1$. Figures 4 and 5 illustrate the case $c$ ($\nabla^2 \psi = \psi + Cz + Dy$) when $R^2(C^2 + D^2) > 1$. Figure 4 shows reversed flow. $C = D = 1, R = 2, A_1 = 50, A_2 = 60$ and $C = D = R = 1, A_1 = 1, A_2 = -1$, respectively, for Figures 4 and 5. The flows when $R^2(C^2 + D^2) = 1$ are given in Figures 6 and 7 when $C = D = 1, R = \frac{1}{\sqrt{2}}, B_1 = 50, B_2 = -60$ and $C = D = 1, R = \frac{1}{\sqrt{2}}, B_1 = 0, B_2 = 1$.

When $R^2(C^2 + D^2) < 1$, we have Figure 8 for $C = D = 1, R = \frac{1}{2}, C_1 = 5, C_2 = 0$. 

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