ABSTRACT. A regular quasi-incomplete locally convex inductive limit of Banach spaces is constructed.

KEY WORDS AND PHRASES. Regular locally convex inductive limit, quasi-completeness, LB-space.


1. INTRODUCTION.

Throughout the paper $E_1 \subset E_2 \subset \cdots$ is a sequence of Hausdorff locally convex spaces with continuous identity maps $E_n \to E_{n+1}$, $n \in \mathbb{N}$. Their locally convex inductive limit is denoted by $\text{ind} E_n$. If all spaces $E_n$ are Banach, resp. Fréchet, then we call $\text{ind} E_n$ an LB-, resp. LF-space.

According to [3], [4 § 5.2], the space $\text{ind} E_n$ is called: $\alpha$-regular if any set bounded in $\text{ind} E_n$ is contained in some $E_n$, $\beta$-regular if any set which is bounded in $\text{ind} E_n$ and contained in some $E_m$ is then bounded in another $E_n$, regular if its is simultaneously $\alpha$-and $\beta$-regular.

By Makarov's Theorem, [4; § 5.6], every Hausdorff quasi-complete LF-space is regular. It is natural to ask whether this theorem can be reversed for LB-spaces. By Raikov's Theorem, [4; § 4.3], every LB-space is quasi-complete iff it is complete. So in [5] Mujica asks: Is every regular LB-space complete? In [6], resp. [7], the authors constructed quasi-, resp. sequentially -, incomplete $\beta$-regular LB-spaces. They erroneously claimed that those spaces were regular. Here we partially correct that error by presenting an example of a regular quasi-incomplete LB-space. The question of existence of a sequentially-incomplete regular LB-space still remains open.

2. NOTATION AND AUXILIARY RESULTS.

Let $N = \{1, 2, 3, \cdots\}, R = (-\infty, \infty)$. Define an order on $N^N$ by $\alpha, \beta \in N^N, \alpha \leq \beta \iff \alpha(n) \leq \beta(n)$ for all $n \in N$. For each $\alpha \in N^N, x \in R^{N \times N}$, and $m, n \in N$, put

$\Gamma(\alpha, x, m) = \sup \{|x_{ij}|; i,j \geq m, j > \alpha(i)\}, a(\alpha)_i = \begin{cases} j^{-1} & \text{if } 1 \leq \alpha(i) < j \leq n \\ 1 & \text{if } i \geq n \end{cases}, (i,j) \in N \times N,$

$X_n = \{x \in R^{N \times N}; \|x\|_n = \sup \{a(\alpha)_i|x_{ij}|; i,j \in N\} < +\infty\},$

$Y_n = \{y \in R^{N \times N}; ||y||_n = \Sigma \{(a(\alpha)_i)^{-1}|y_{ij}|; i,j \in N\} < +\infty\},$

$E_n = \{x \in X_n; \lim_{m \to \infty} \Gamma(\alpha, x, m) = 0 \text{ for some } \alpha \in N^N\}.$
For brevity we write $X = \text{ind}X_n, Y = \text{proj}Y_n, E = \text{ind}E_n$. Finally, we have an inner product $(x, y) \mapsto x \cdot y = \sum_{i,j \in N} x_{ij} y_{ij}$ defined on $X \times Y$, $n \in N$, and on $X \times Y$.

**Lemma 1.** For any sequence $\{\alpha_k; k \in N\} \subset N^N$ there exists $\alpha \in N^N$ such that $\liminf_{m \to \infty} \alpha_k(m) \geq 1$ for all $k \in N$.

**Proof.** Put $\alpha(m) = \max\{\alpha_k(m); k \leq m\}, m \in N$. Then $\alpha = (\alpha(1), \alpha(2), \cdots)$ has the required property.

**Lemma 2.** For each $n \in N$:

(a) $X_n, Y_n$ are Banach spaces.

(b) $E_n$ is a closed subspace of $X_n$. Hence it is also a Banach space.

(c) $X_n \subset X_{n+1}, Y_n \supset Y_{n+1}$, and $E_n \subset E_{n+1}$, where all inclusions are continuous.

**Proof.** (a) Each $X_n$, resp. $Y_n$, as a weighted $l^\infty$, resp. $l^1$-space, is Banach.

(b) If $x_1, x_2 \in E_n$, there are $\alpha_1, \alpha_2 \in N^N$ such that $\lim_{m \to \infty} \Gamma(\alpha_1, x_1, m) = 0, \ i = 1, 2$. Then we have $\lim_{m \to \infty} \Gamma(\alpha_1 + \alpha_2, x_1 + x_2, m) = 0$. Hence $x_1 + x_2 \in E_n$ and $E_n$ is a linear subspace of $X_n$.

Let $\{x(k); k \in N\}$ be a sequence in $E_n$ with a limit $x \in X_n$. For each $k \in N$ take $\alpha_k \in N^N$ for which $\lim_{m \to \infty} \Gamma(\alpha_k, x(k), m) = 0$. By Lemma 1, there is $\alpha \in N^N$ such that $\liminf_{m \to \infty} \alpha_k(m) \geq 1$ for any $k \in N$.

Given an arbitrary $\varepsilon > 0$, choose $k \in N$ so that $\|x - x(k)\|_n < \varepsilon$. For this particular $k$, take $m_1, m_2 \in N$ so that $\frac{\alpha(m)}{\alpha_k(m)} > \frac{1}{2}$ for any $m \geq m_1$, and $\Gamma(\alpha_k, x(k), m) < \varepsilon$ for any $m \geq m_2$. Finally, put $m_0 = \max\{m_1, m_2, n\}$. If $m \geq m_0$ then for $i,j \geq m, \ j > 2\alpha(i)$, we have $j > \alpha_k(i)$ which implies $|x(k)_{ij}| \leq \Gamma(\alpha_k, x(k), m)$. Moreover $\alpha_n(1) = 1$ since $i \geq n$. Hence $|x_{ij}| = \alpha_n(1) |x_{ij}| \leq \alpha_n(1)(|x_{ij} - x(k)_{ij}| + |x(k)_{ij}|) \leq \|x - x(k)\|_n + \Gamma(\alpha_k, x(k), m) < \varepsilon + \varepsilon$. Thus $\Gamma(2\alpha, x, m) < 2\varepsilon$ and $x \in E_n$.

(c) For each $(i, j) \in N \times N$, we have $\alpha_n + 1, j \leq \alpha_n(j)$. Hence $\|x\|_{n+1} \leq \|x\|_n$ for any $x \in X_n$ and $\||y||_{n+1} \leq \||y||_n$ for any $y \in Y_{n+1}$.

**Lemma 3.** For each $n \in N$, let $e_n > 0, B_n = \{x \in E_n; \|x\|_n < e_n\}$, and $V$ be the convex hull of $U(B_n; n \in N)$. Then the closure $\overline{V}$ of $V$ in $E$ is the same as the $\sigma(E, Y)$-closure of $V$.

**Proof.** Let $E'$ be the dual space for $E$. From the duality theory we know that $\overline{V}$ is the same as the $\sigma(E, E')$-closure of $V$. Since $Y \subset E'$, we have $\sigma(E, Y) \subset \sigma(E, E')$. Thus it remains to show that if $v \in E$ is a $\sigma(E, Y)$-limit of a net $\alpha \mapsto v(\alpha) : A \to V$, then $v$ is in the $\sigma(E, E')$-closure of $V$.

For each $\alpha \in A$, there exists $m(\alpha) \in N$ such that $v(\alpha) = \Sigma(\lambda(\alpha, p) b(\alpha, p); p = 1, 2, \cdots, m(\alpha))$, where $\lambda(\alpha, p) > 0, \Sigma(\lambda(\alpha, p); p = 1, 2, \cdots, m(\alpha)) = 1$, and $b(\alpha, p) \in B_{n(\alpha, p), 1} \leq n(\alpha, 1) < n(\alpha, 2) < \cdots < n(\alpha, m(\alpha))$. Take $(i, j) \in N \times N$. Let $r$ be the largest integer, less than or equal to $m(\alpha)$, for which $S_r = \Sigma(\lambda(\alpha, p) b(\alpha, p); p = 1, 2, \cdots, m(\alpha))$. Denote the signum function by $\text{sgn}$ and put

$$c(\alpha, p) = \begin{cases} (\text{sgn} v_{ij}) |b(\alpha, p)|, & p \leq r \\ \left|\lambda(\alpha, p + 1)\right|^{-1} (\text{sgn} v_{ij}) |v_{ij} - S_r|, & p = r + 1, m(\alpha) \\ 0, & r + 1 < p \leq m(\alpha) \end{cases}$$

Then $|c(\alpha, p)_{ij}| \leq |b(\alpha, p)_{ij}|$ for each $p \leq m(\alpha)$ which implies $c(\alpha, p) \in B_{n(\alpha, p), p = 1, 2, \cdots, m(\alpha)} \subset V$. Moreover

(1) $|w(\alpha)_{ij}| \leq |v_{ij}|$.

(2) $|v_{ij} - w(\alpha)_{ij}| \leq |v_{ij} - w(\alpha)_{ij}|$.

To prove (1) and (2), we have to distinguish two cases:

(a) $r < m(\alpha)$. Then $|w(\alpha)_{ij}| \leq \Sigma(\lambda(\alpha, p) |c(\alpha, p)|_{ij}; p = 1, 2, \cdots, r + 1) = |v_{ij}|$ and $|v_{ij} - w(\alpha)_{ij}| = (\text{sgn} v_{ij})(|v_{ij} - w(\alpha)_{ij}|) = |v_{ij} - \Sigma(\lambda(\alpha, p) |c(\alpha, p)|_{ij}; p = 1, 2, \cdots, r + 1) = 0 = |v_{ij} - w(\alpha)_{ij}|$. 

(b) $r \geq m(\alpha)$. Then $|w(\alpha)_{ij}| \leq \Sigma(\lambda(\alpha, p) |c(\alpha, p)|_{ij}; p = 1, 2, \cdots, m(\alpha)) \leq m(\alpha) = |v_{ij}|$ and $|v_{ij} - w(\alpha)_{ij}| = (\text{sgn} v_{ij})(|v_{ij} - w(\alpha)_{ij}|) = |v_{ij} - \Sigma(\lambda(\alpha, p) |c(\alpha, p)|_{ij}; p = 1, 2, \cdots, m(\alpha)) = 0 = |v_{ij} - w(\alpha)_{ij}|$.
\( r = m(\alpha) \). Then \( \| w(\alpha)_{ij} \| \leq \sum \{ (\lambda(\alpha, p) c(\alpha, \beta)_i) ; p = 1, 2, \ldots, m(\alpha) \} \leq \sum \{ (\lambda(\alpha, p) b(\alpha, \beta)_i) ; p = 1, 2, \ldots, m(\alpha) \} \leq \| w_{ij} \| \) and \( \| w_{ij} - w_{ij}(\alpha) \| = \| w_{ij} \| - \sum \{ \lambda(\alpha, p) b(\alpha, \beta)_i ; p = 1, 2, \ldots, m(\alpha) \} = \| w_{ij} - v(\alpha)_{ij} \| \).

The Banach space \( c_0(N \times N) \) of double null sequences is contained in \( E \) and the identity maps \( x \mapsto x \mapsto x : c_0(N \times N) \to E_1 \to E \) are continuous. Hence the restriction of each \( f \in E' \) to \( c_0(N \times N) \) is continuous. It follows from the Riesz-Kakutani-Hewitt Representation Theorem that there exists a signed, regular, bounded, Borel measure \( \mu \) on the discrete locally compact Hausdorff space \( N \times N \) such that \( f(x) = \int x d\mu, x \in c_0(N \times N) \).

Each \( x \in E \) is a pointwise limit, as well as a limit in \( E \), of a sequence \( \{ x(k) \in c_0(N \times N) ; k \in N \} \) satisfying \( |x(k)_{ij}| \leq |x_{ij}|, i, j, k \in N \). Hence it follows from the Lebesgue Dominant Convergence Theorem that \( f(x(k)) = \int x(k) d\mu \to \int x d\mu. \) Since \( f(x(k)) \to f(x) \), we have \( f(x) = \int x d\mu, x \in E \).

The \( \sigma(E, Y) \)-convergence implies the pointwise convergence. Thus, according to (2), \( w(\alpha) \to v \) pointwise. Then, by (1) and the Lebesgue Dominant Convergence Theorem, we have \( f(w(\alpha)) = \int w(\alpha) d\mu \to \int v d\mu = f(v), f \in E', v \in E' \), and \( v \) is in the \( \sigma(E, E') \)-closure of \( V \).

**LEMMA 4.** Let \( \tilde{V} \) be the same closed neighborhood of 0 in \( E \) as in Lemma 3 and for each \( \alpha \in N^N \), \( (i, j) \in N \times N \),

\[
\begin{aligned}
\{ 1 \text{ if } j \leq \alpha(i) \text{ and } j = 2^k \text{ for some } k \in N \\
0 \text{ otherwise}
\}
\end{aligned}
\]

Then \( x(\alpha) \in E_1, \| x(\alpha) \|_1 = 1 \), and there exists \( \gamma \in N^N \) such that \( x(\alpha) - x(\beta) \in \tilde{V} \) for any \( \alpha, \beta \geq \gamma \).

**PROOF.** Clearly \( \| x(\alpha) \|_1 = 1 \) and \( \Gamma(\alpha, x(\alpha), m) = 0 \) for any \( \alpha \in N^N, m \in N \). Hence \( \lim_{m \to \infty} \Gamma(\alpha, x(\alpha), m) = 0 \) and the first statement holds.

Let \( V_0 = \{ y \in Y; |y|, x > |x|, x \in \tilde{V} \} \). Then the polar \( (V_0)^0 \) in \( E \) is the \( \sigma(E, Y) \)-closure of \( V \) which, by the Lemma 3, equals \( \tilde{V} \). The polars \( V^0 \) and \( V_0 \) in \( (E', \sigma(E', E)) \) are equal. Hence \( V^0 = V_0 \) which implies that \( V_0 = (V_0)^0 \) dense in \( V_0 \). Thus to prove that \( x(\alpha) - x(\beta) \in (V_0) \) holds, it suffices to show \( |y, x(\alpha) - x(\beta)| \leq 1 \) for all \( y \in V_0 \).

Choose \( \gamma \in N^N \) so that \( \gamma(n) > n + 4^{a^2} \), \( n \in N \), and an arbitrary \( y \in V_0 \). Denote by \( |y| \) the element of \( Y \) defined by \( |y| = |y|, (i, j) \in N \times N \). Since \( V \) is a balanced set, we have \( |y| \in V_0 \).

For each \( n \in N \), put

\[
d(n)_{ij} = \begin{cases} \sqrt{j} & \text{if } i = n, j > \gamma(n), j = 2^k \text{ for some } k \in N \\
0 & \text{otherwise}
\end{cases}
\]

Then \( \| d(n) \|_n \leq (\gamma(n))^{\frac{1}{2}} \leq \varepsilon_n. \) Hence \( d(n) \in B_n \) and \( |y|, d(n) > |y| \) \( \leq 1 \). Finally, for \( \alpha, \beta \geq \gamma \), we have \( |y, x(\alpha) - x(\beta)| = |\sum \{ y_{ij}(x(\alpha)_{ij} - x(\beta)_{ij}) ; (i, j) \in N \times N \}| \leq \sum \{ |y_{ij}| (x(\alpha)_{ij} - x(\beta)_{ij}) ; i, j \in N \} \leq \sum \{ |y_{ij}| 2^k > \gamma(i) \} \}

3. MAIN RESULTS.

**PROPOSITION 1.** The net (3) is bounded in \( E_1 \) and Cauchy in \( E \).

**Proof follows from Lemma 4.**

**PROPOSITION 2.** The net (3) does not converge in \( E \).

**PROOF.** Assume \( x(\alpha) \to x \) in \( E \). For each \( (i, j) \in N \times N \) the functional \( z \mapsto z_{ij} : E \to R \) is continuous. It implies \( x(\alpha)_{ij} \to z_{ij} \). Fix \( (i, j) \in N \times N \) and choose \( \gamma \in N^N \) so that \( \gamma(i) \geq j \). Then
for $\alpha \geq \gamma$, we have

$$x(\alpha)_{ij} = x(\gamma)_{ij} = \begin{cases} 1 & \text{if } j = 2^k \text{ for some } k \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}.$$  

Take $\alpha \in \mathbb{N}$ and $m \in \mathbb{N}$. Then for $i \geq m, 2^k > \alpha(i)$, we have $1 = x_{i,2^k} \leq \Gamma(\alpha, x, m)$. Hence $x \not\in E_n$ for any $n \in \mathbb{N}$.

**PROPOSITION 3.** The space $E$ is regular.

**PROOF.** Assume that $E$ is not regular. Then there exists a set $B$ bounded in $E$ such that for any $n \in \mathbb{N}$ either $B \not\in E_n$ or $B \not\in E_n$. Choose $z(1) \in B$, $z(1) \neq 0$, and $(i(1), j(1)) \in \mathbb{N} \times \mathbb{N}$ so that $z(1)_{i(1),j(1)} \neq 0$. Put $\varepsilon_1 = |z(1)_{i(1),j(1)}|$. Suppose that $z(k), i(k), j(k)$, and $\varepsilon_k, k = 1, 2, \ldots, n - 1$, where $n > 1$, have been selected. Then there are two cases: Either $B \subset E_n$ and $B$ is not bounded in $E_n$ or there exists $x \in B \setminus E_n$. In the second case $\|x\|_n = +\infty$. Hence in either case there is $z(n) \in B$ such that $\|z(n)\|_n > n \cdot \max\{\varepsilon_k; k = 1, 2, \ldots, n - 1\}$ and we can choose $(i(n), j(n)) \in \mathbb{N} \times \mathbb{N}$ so that

$$a_n(i(n), j(n))z(n)_{i(n), j(n)} \geq n \cdot \max\{\varepsilon_k; k = 1, 2, \ldots, n - 1\}.$$  

Choose $z(n) \in B$ such that $\|z(n)\|_n > n \cdot \max\{\varepsilon_k; k = 1, 2, \ldots, n - 1\}$ and we can choose $(i(n), j(n)) \in \mathbb{N} \times \mathbb{N}$ so that $\|z(n)\|_n > n \cdot \max\{\varepsilon_k; k = 1, 2, \ldots, n - 1\}$. Put $\varepsilon_{n+1} = \min\{z(n)_{i(n), j(n)}; k = 1, 2, \ldots, n\}$. Then

$$\varepsilon_{n+1} \leq \frac{1}{n}a(p)_{i(r), j(r)}|z(r)_{i(r), j(r)}|$$  

for any $p, r \in \mathbb{N}$.

In fact, for $p \geq r$ the inequality (6) follows from (5) and for $p < r$ the inequality (4) implies

$$\varepsilon_{n+1} \leq \frac{1}{n}a(r)_{i(r), j(r)}|z(r)_{i(r), j(r)}| \leq \frac{1}{n}a(p)_{i(r), j(r)}|z(r)_{i(r), j(r)}|.$$  

Let a 0-neighborhood $V$ be the same as in Lemma 3. Since $B$ is bounded in $E$ there exists $r \in \mathbb{N}$ such that $B \subset rV$. Hence also $z(r) \in rV$ and $z(r) = r\Sigma\{\lambda_py(p); p = 1, 2, \ldots, s\}$, where $\lambda_p \geq 0, \Sigma\{\lambda_p; p = 1, 2, \ldots, s\} = 1$ and $y(p) \in B_p$. By (6), we have $a(p)_{i(r), j(r)}|y(p)_{i(r), j(r)}| \leq \frac{1}{n}a(p)_{i(r), j(r)}|z(r)_{i(r), j(r)}|$, which implies $|y(p)_{i(r), j(r)}| \leq \frac{1}{n}|z(r)_{i(r), j(r)}|$, $p = 1, 2, \ldots, s$. Hence $|z(r)_{i(r), j(r)}| = |r\Sigma\{\lambda_py(p)_{i(r), j(r)}; p = 1, 2, \ldots, s\}| \leq r\Sigma\{|\lambda_py(p)_{i(r), j(r)}|; p = 1, 2, \ldots, s\} < r\Sigma\{|\lambda_py(r)_{i(r), j(r)}|; p = 1, 2, \ldots, s\}$, a contradiction.

By combining all three Propositions we get:

**THEOREM.** The space $\text{ind}E_n$ is a regular LB-space which is not quasi-complete.

**REFERENCES**