THE NACHBIN COMPACTIFICATION VIA CONVERGENCE ORDERED SPACES

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ABSTRACT. We construct the Nachbin compactification for a $T_{3.5}$-ordered topological ordered space by taking a quotient of an ordered convergence space compactification. A variation of this quotient construction leads to a compactification functor on the category of $T_{3.5}$-ordered convergence ordered spaces.

KEY WORDS AND PHRASES: topological ordered space, convergence ordered space, $T_1$-ordered space, $T_{3.5}$-ordered space, Nachbin compactification.

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0. INTRODUCTION.

The Nachbin (or Stone-Cech-ordered) compactification (see [1], [6]) is the largest $T_2$-ordered topological ordered compactification of a $T_{3.5}$-ordered topological ordered space. In [4], one of the authors and G.D. Richardson constructed an ordered compactification $(X^*, C)$ for an arbitrary convergence ordered space $X$. This latter compactification exhibits essentially the same universal property as the Nachbin compactification, but behaves poorly relative to separation properties (see Example 1.4).

Starting with an arbitrary convergence ordered space $X$, we introduce an equivalence relation $\mathcal{R}$ on the set $|X^*|$ which underlies $X^*$, and obtain an ordered quotient space $X^*/\mathcal{R}$ which is both compact and $T_2$-ordered. We next give two conditions $C$ and $O$ which are necessary and sufficient to make the natural map from $X$ into $X^*/\mathcal{R}$ both an order embedding and a homeomorphic embedding, so that $X^*/\mathcal{R}$ becomes a $T_2$-ordered convergence ordered compactification of $X$. For ordered convergence spaces $X$ satisfying conditions $C$ and $O$, it turns out that the topological modification $\lambda X$ of $X$ is a $T_{3.5}$-ordered topological ordered space, and $\lambda(X^*/\mathcal{R})$ is the Nachbin compactification of $\lambda X$. In particular, if $X$ is assumed to be a $T_{3.5}$-ordered topological ordered space, then $\lambda(X^*/\mathcal{R})$ is the Nachbin compactification of $X$.

In addition to giving an alternate construction for the Nachbin compactification, we obtain some interesting results pertaining to convergence ordered compactifications. In Section 3, we define a
regular convergence ordered space satisfying conditions $C$ and $O$ to be a $T_{3,5}$-ordered convergence ordered space, and we show that for such a space $X$, the regular modification $r(X^*/R)$ of the quotient $X^*/R$ is a regular, $T_2$-ordered convergence ordered compactification of $X$. Relative to this compactification functor, the regular, $T_2$-ordered, compact convergence spaces (with increasing, continuous maps as morphisms) form an epireflective subcategory of the category of all $T_{3,5}$-ordered convergence ordered spaces (with increasing, continuous maps as morphisms).

1. PRELIMINARIES.

We introduce some basic notation and terminology and summarize some results from [4]. If $(X, \leq)$ is a poset, and $A \subseteq X$, we denote by $i(A), d(A), \text{ and } A^\wedge$ the increasing, decreasing, and convex hulls, respectively, of $A$; note that $A^\wedge = i(A) \cap d(A)$. Similarly, if $F(X)$ is the set of all (proper) filters on $X$ and $\mathcal{F} \in F(X)$, let $i(\mathcal{F})$, the filter generated by $\{i(F) : F \in \mathcal{F}\}$, be the increasing hull of $\mathcal{F}$; the decreasing hull $d(\mathcal{F})$ and convex hull $\hat{\mathcal{F}}$ are defined analogously. A filter $\mathcal{F}$ is said to be convex if $\mathcal{F} = \hat{\mathcal{F}}$. Note that $\hat{\mathcal{F}} = i(\mathcal{F}) \vee d(\mathcal{F})$.

If $(X, \leq, \rightarrow)$ is a poset $(X, \leq)$ equipped with a convergence structure $\rightarrow$ which is locally convex (i.e., $f \rightarrow z$ whenever $\mathcal{F} \rightarrow z$), then $(X, \leq, \rightarrow)$ is called a convergence ordered space; we usually write $X$ rather than $(X, \leq, \rightarrow)$ when there is no danger of ambiguity. A convergence ordered space is $T_1$-ordered if the sets $i(x)$ and $d(x)$ are closed for all $x \in X$, and $T_2$-ordered if the order $\leq$ is a closed subset of $X \times X$. For any convergence ordered space $X$, let $CI'(X)$ (respectively, $CD'(X)$) denote the set of all continuous, increasing (respectively, decreasing) maps from $X$ into $[0,1]$.

A convergence ordered space whose convergence structure is a topology is called a topological ordered space. Such a space is said to be convex if the open monotone (i.e., increasing or decreasing) sets form a subbase for the topology. For the remainder of this paper, we shall adopt the notational abbreviation used in [4] and write "t.o.s" instead of "topological ordered space" and "c.o.s." in place of "convergence ordered space".

A t.o.s. $X$ is said to be $T_{3,5}$-ordered if it satisfies the following conditions: (1) If $x \in X, A$ is a closed subset of $X$, and $x \not\in A$, then there is $f \in CI'(X)$ and $g \in CD'(X)$ such that $f(x) = g(x) = 0$ and $f(y) \lor g(y) = 1$, for all $y \in A$; (2) If $x \not\leq y$ in $X$, there is $f \in CI'(X)$ such that $f(y) = 0$ and $f(x) = 1$. The $T_{3,5}$-ordered spaces are precisely those which allow $T_2$-ordered t.o.s. compactifications, and all $T_{3,5}$-ordered spaces are convex.

If $X$ is a $T_{3,5}$-ordered t.o.s., then the Nachbin compactification of $X$ (see [1], [6]) is obtained by embedding $X$ into an "ordered cube", whose component intervals are indexed by $CI'(X)$. The Nachbin compactification $\beta_0X$ is characterized by the following well-known result.

PROPOSITION 1.1. If $X$ is a $T_{3,5}$-ordered t.o.s., then $\beta_0X$ is $T_2$-ordered. Furthermore, if $f : X \rightarrow Y$ is an increasing, continuous map and $Y$ is a compact, $T_2$-ordered t.o.s., then $f$ has a unique, increasing, continuous extension $f' : \beta_0X \rightarrow Y$.

We next describe briefly the construction of the convergence ordered compactification $X^*$ of an arbitrary c.o.s. $X$ described in [4], which has essentially the same lifting property as $\beta_0X$. Given a c.o.s. $X$, let $X'$ be the set of all non-convergent maximal convex filters on $X$, and let $X^* = \{z : z \in X\} \cup X'$. Before proceeding further, it will be useful to establish the following proposition about maximal convex filters.

PROPOSITION 1.2. The maximal convex filters on a poset $X$ are precisely the set $\{\hat{\mathcal{F}} : \mathcal{F} \text{ is an ultrafilter on } X\}$.

PROOF. Clearly every maximal convex filter is the convex hull of every finer ultrafilter. Conversely, suppose $\mathcal{F}$ is an ultrafilter on $X$ and $\mathcal{G}$ is a convex filter such that $\hat{\mathcal{F}} \leq \mathcal{G}$. Then for any
convex set $G \in \mathcal{G}$, the filters $\mathcal{F}_1$ and $\mathcal{F}_2$ generated by \{i($G$) $\cap$ $F : F \in \mathcal{F}$\} and \{d($G$) $\cap$ $F : F \in \mathcal{F}$\}, respectively, are well-defined filters finer than, and hence equal to, $\mathcal{F}$. Thus i($G$) $\in \mathcal{F}$ and d($G$) $\in \mathcal{F}$ implies i($G$) $\cap$ d($G$) $\in \mathcal{F}$; therefore $\mathcal{G} = \mathcal{F}$.

Again assuming that $X$ is an arbitrary c.o.s., let $\varphi : X \to X^*$ be defined by $\varphi(x) = \dot{x}$, for all $x \in X$. A partial order $\leq^*$ is defined on $X^*$ as follows: $\mathcal{F} \leq^* \mathcal{G}$ iff i($\mathcal{F}$) $\leq \mathcal{G}$ (or, equivalently, d($\mathcal{G}$) $\leq \mathcal{F}$). Since $x \leq y$ iff $\dot{x} \leq^* \dot{y}$, $\varphi : (X, \leq) \to (X^*, \leq^*)$ is an order embedding.

If $A \subseteq X$, let $A^* = \{\mathcal{F} \in X^* : A \in \mathcal{F}\}$; if $\mathcal{F} \in \mathcal{F}(X)$, let $\mathcal{F}^*$ denote the filter in $\mathcal{F}(X^*)$ generated by $\{F^* : F \in \mathcal{F}\}$. A convergence structure $\rightarrow$ on $(X^*, \leq^*)$ is defined as follows: For $A \in \mathcal{F}(X^*)$,

$A \rightarrow \dot{x} \in \varphi(X)$ iff there is $\mathcal{F} \to x$ such that $\mathcal{F}^* \leq A$;

$A \rightarrow \mathcal{G} \in X^*$ iff $\mathcal{G}^* \leq A$.

Writing $X^*$ in place of $(X^*, \leq^*, \rightarrow)$, we state the following result which is proved in [4].

**PROPOSITION 1.3.** If $X$ is a c.o.s., then $(X^*, \varphi)$ is a convergence ordered compactification of $X$. If $f : X \to Y$ is a continuous, increasing map and $Y$ a compact, regular, $T_2$-ordered c.o.s., then $f$ has a unique, increasing, continuous extension $f : X^* \to Y$.

Recall that a convergence space $Y$ is regular if $\text{cl}_Y \mathcal{F} \to x$ whenever $\mathcal{F} \to x$. Here "\text{cl}_Y" is the closure operator for $Y$, and $\text{cl}_Y \mathcal{F}$ is the filter on $Y$ generated by $\{cl_Y F : F \in \mathcal{F}\}$.

In [4], a c.o.s. $X$ is defined to be strongly $T_2$-ordered if $X$ is $T_2$ (i.e., convergent filters have unique limits) and the following conditions hold: (S1) if $\mathcal{F} \to x, \mathcal{G} \in X'$, and i($\mathcal{F}$) $\leq \mathcal{G}$, then d($\mathcal{G}$) $\leq \dot{x}$; (S2) if $\mathcal{F} \to x, \mathcal{G} \in X'$, and d($\mathcal{F}$) $\leq \mathcal{G}$, then i($\mathcal{G}$) $\leq \dot{x}$. In Proposition 2.8, [4], it is shown that $X^*$ is $T_2$-ordered iff $X$ is strongly $T_2$-ordered. As we see in the next example, very nice c.o.s.'s may fail to be strongly $T_2$-ordered.

**EXAMPLE 1.4.** Let $X$ be the Euclidean plane with its usual (product) order and topology. Let $\mathcal{F}$ be the filter on $X$ generated by sets of the form $F_\alpha = \{(a, b) \in X : -\alpha < a < 0, \ b = 0\}$ for each natural number $n$, and let $x = (0, 0)$. Let $\mathcal{G}$ be the convex hull of any ultrafilter containing the set $S = \{(a, b) \in X : a = -b^{-1}\}$ and coarser than the filter $\mathcal{H}$ generated by sets of the form $H_\alpha = \{(a, b) \in X : b \geq n\}$ for $n = 1, 2, 3, \ldots$. Note that (S1) is violated by $\mathcal{F}, \mathcal{G}$ and $x$; thus the compactification $X^*$ of $X$ is not $T_2$-ordered.

2. $\beta_0 X$ AS A QUOTIENT OF $X^*$.

Let $(X, \leq, \rightarrow)$ be any c.o.s., and let $(X^*, \varphi)$ be the convergence ordered compactification of $X$ constructed in the last section. By Proposition 1.3 there is, for any $f \in CI(X^*)$, a unique, continuous, increasing extension $f_\varphi : X^* \to [0, 1]$.

We define an equivalence relation $\mathcal{R}$ on $X^*$ as follows: $\mathcal{R} = \{\mathcal{F}, \mathcal{G} \in X^* \times X^* : f_\varphi(\mathcal{F}) = f_\varphi(\mathcal{G}), \text{ for all } f \in CI(X^*)\}$. Let $\sigma$ be the projection map of $X^*$ onto $X^*/\mathcal{R}$ (i.e., for each $\mathcal{F} \in X^*$, $\sigma(\mathcal{F}) = [\mathcal{F}]$, where $[\mathcal{F}]$ is the $\mathcal{R}$-equivalence class containing $\mathcal{F}$). A partial order $\leq_\mathcal{R}$ on $X^*/\mathcal{R}$ is defined as follows:

$[\mathcal{F}] \leq_\mathcal{R} [\mathcal{G}]$ iff $f_\varphi([\mathcal{F}]) \leq f_\varphi([\mathcal{G}])$ in $R$ for all $f \in CI(X^*)$.

We also impose on $X^*/\mathcal{R}$ the quotient convergence structure which is described (see [2]) as follows: If $\Phi \in \mathcal{F}(X^*/\mathcal{R})$ and $[\mathcal{F}] \in X^*/\mathcal{R}$, then $\Phi \to [\mathcal{F}]$ in $X^*/\mathcal{R}$ iff there is $\mathcal{F} \in [\mathcal{F}]$ and there is a filter $A \in \mathcal{F}(X^*)$ such that $A \rightarrow \mathcal{F}$ in $X^*$ and $\sigma(A) \leq \Phi$.

**THEOREM 2.1.** For any c.o.s. $X$, $X^*/\mathcal{R}$ is a compact, $T_2$-ordered c.o.s.
PROOF. $X'/\mathcal{R}$ is obviously compact. To show that $X'/\mathcal{R}$ is $T_2$-ordered, it is sufficient (by Proposition 1.2, [4]) to show that if $\Phi, \Theta \in F(X'/\mathcal{R})$, $\Phi \rightarrow [\mathcal{F}]$ and $\Theta \rightarrow [G]$ in $X'/\mathcal{R}$, and $\Phi \otimes \Theta$ has a trace on the order $\leq$, then $[\mathcal{F}] \leq_k [G]$.

If $f \in C\Phi(X)$, define $f_{\Phi} : X'/\mathcal{R} \rightarrow [0,1]$ by $f_{\Phi}(f) = f_{\Phi}(f)$, for all $f \in X'$. It is easy to verify that $f_{\Phi}$ is well-defined and $f_{\Phi} \in C\Phi(X'/\mathcal{R})$. If $\Phi \rightarrow [\mathcal{F}]$ and $\Theta \rightarrow [G]$ in $X'/\mathcal{R}$ and $\Phi \otimes \Theta$ has a trace on $\leq$, it follows that $f_{\Phi}(\Phi) \times f_{\Theta}(\Theta)$ has a trace on the order of $[0,1]$; since $[0,1]$ is $T_2$-ordered, $f_{\Phi}(f) = f_{\Phi}(f) \leq f_{\Phi}(f) = f_{\Phi}(f)$. The latter inequality holds for all $f \in C\Phi(X)$, and so $[\mathcal{F}] \leq_k [G]$, which establishes that $X'/\mathcal{R}$ is $T_2$-ordered.

For an arbitrary c.o.s. $X$, we have already defined the continuous, increasing maps $\phi : X \rightarrow X'$ and $\sigma : X' \rightarrow X'/\mathcal{R}$; we define $\varphi_{\mathcal{R}} : X \rightarrow X'/\mathcal{R}$ by $\varphi_{\mathcal{R}} = \sigma \circ \phi$. It is clear that $\varphi_{\mathcal{R}}(X)$ is dense in the compact, $T_2$-ordered c.o.s. $X'/\mathcal{R}$. We are now interested in characterizing those spaces $X$ for which $(X'/\mathcal{R}, \varphi_{\mathcal{R}})$ is a compactification. With this goal in mind, we introduce the following conditions.

CONDITION C. For any maximal convex filter $\mathcal{F}$ on $X$, $\mathcal{F} \rightarrow x$ in $X$ iff $f(\mathcal{F}) \rightarrow f(x)$ in $[0,1]$ for all $f \in C\Phi(X)$.

CONDITION O. For any points $x, y$ in $X$, $x \leq y$ in $X$ iff $f(x) \leq f(y)$ in $[0,1]$, for all $f \in C\Phi(X)$.

It is easy to verify that any $T_2$-ordered t.o.s. satisfies Conditions C and O.

LEMMA 2.2. If $X$ is a c.o.s. satisfying Conditions C and O, then $[\tilde{x}] = \{\tilde{x}\}$, for all $x \in X$.

PROOF. $C\Phi(X)$ separates points in $X$ by Condition O, and so $\sigma$ is one-to-one on $\varphi(X)$. This implies $\tilde{y} \notin [\tilde{x}]$ if $x \neq y$. Next, assume that there is $\mathcal{F} \in \mathcal{X}' \cap [\tilde{x}]$. Then $f_{\Phi}(\mathcal{F}) = f_{\Phi}(\tilde{x}) = f(\tilde{x})$ for all $f \in C\Phi(X)$; in other words, $f(\mathcal{F}) \rightarrow f(x)$ in $[0,1]$. This implies $\sigma(\mathcal{F}) = \sigma(\tilde{x}) = f(x)$ for all $f \in C\Phi(X)$. Condition C then implies $\mathcal{F} \rightarrow x$ in $X$, contradicting the assumption $\mathcal{F} \in \mathcal{X}'$.

THEOREM 2.3. Let $X$ be a c.o.s. Then $\varphi_{\mathcal{R}} : X \rightarrow X'/\mathcal{R}$ is an order and a homeomorphic embedding iff $X$ satisfies Conditions C and O.

PROOF. Suppose that $X$ satisfies Conditions C and O. Then $\varphi_{\mathcal{R}}$ is one-to-one since $C\Phi(X)$ separates points in $X$. Also note that $\varphi_{\mathcal{R}} = \sigma \circ \varphi = (\sigma|_{\varphi(X)}) \circ \varphi$, and thus $\sigma|_{\varphi(X)}$ is one-to-one.

Let $\Phi \rightarrow [\tilde{x}]$ in $X'/\mathcal{R}$. Then there is $\mathcal{A} \in F(X')$ such that $\mathcal{A} \rightarrow \tilde{x}$ in $X'$ and $\Phi \geq \sigma(\mathcal{A})$. By definition of $*$ convergence in $X'$, there is a filter $\mathcal{F}$ on $X$ such that $\mathcal{F} \rightarrow x$ and $\mathcal{A} \geq \mathcal{F}$. Therefore, $\varphi^{-1}_{\mathcal{R}}(\Phi) \geq \varphi^{-1}_{\mathcal{R}}(\sigma(\mathcal{A})) \geq \varphi^{-1}_{\mathcal{R}}(\sigma(\mathcal{F})) = \varphi^{-1} \cdot (\sigma|_{\varphi(X)}^{-1}(\sigma(\mathcal{F})))$. It follows by Lemma 2.2 that $(\sigma|_{\varphi(X)}^{-1}(\sigma(\mathcal{F}))) \geq \mathcal{F}$. Consequently, $\varphi^{-1}_{\mathcal{R}}(\Phi) \geq \varphi^{-1}(\mathcal{F}) = \mathcal{F} \rightarrow x = \varphi^{-1}(\tilde{x})$, i.e. $\varphi^{-1}_{\mathcal{R}}(\Phi) \rightarrow \varphi^{-1}(\tilde{x})$. Thus $\varphi^{-1}_{\mathcal{R}}$ is continuous.

Let $[\tilde{x}] \leq_k [\tilde{y}]$ in $X'/\mathcal{R}$; then for any $f \in C\Phi(X)$, $f_{\Phi}(\tilde{x}) \leq f_{\Phi}(\tilde{y})$, i.e. $f_{\Phi}(\varphi(x)) \leq f_{\Phi}(\varphi(y))$, which implies $f(x) \leq f(y)$ for all $f \in C\Phi(X)$. By Condition O, $x \leq y$. Thus $\varphi^{-1}_{\mathcal{R}}$ is increasing, and we conclude that $\varphi_{\mathcal{R}}$ is an order and homeomorphic embedding.

Conversely, assume that $\varphi_{\mathcal{R}}$ is both an order and homeomorphic embedding. Let $\mathcal{F}$ be a maximal convex filter on $X$ such that, for some $x \in X$, $f(\mathcal{F}) \rightarrow f(x)$ for all $f \in C\Phi(X)$. Suppose $\mathcal{F} \rightarrow x$ is not true. Then we need to consider two cases.

CASE 1. $\mathcal{F} \rightarrow y$ and $y \neq x$. This implies that for each $f \in C\Phi(X)$, $f(\mathcal{F}) \rightarrow f(y)$. From this we deduce that $[\tilde{x}] = [\tilde{y}]$, which is a contradiction, since $\varphi_{\mathcal{R}}$ is assumed to be one-to-one.

CASE 2. $\mathcal{F} \in X'$. This leads to the conclusion that $[\mathcal{F}] = [\tilde{x}]$; in other words, $\varphi_{\mathcal{R}}(\mathcal{F}) \rightarrow [\tilde{x}]$ in $X'/\mathcal{R}$, which implies $\mathcal{F} \rightarrow x$ in $X$, since $\varphi_{\mathcal{R}}$ is a homeomorphic embedding. This contradicts $\mathcal{F} \in X'$. We therefore conclude that $X$ satisfies Condition C.

Finally, let $x, y \in X$ such that $f(x) \leq f(y)$ for all $f \in C\Phi(X)$. Then $f_{\Phi}(\varphi(x)) \leq f_{\Phi}(\varphi(y))$ for all $f \in C\Phi(X)$, i.e. $f_{\Phi}(\tilde{x}) \leq f_{\Phi}(\tilde{y})$ for all $f \in C\Phi(X)$. This implies $[\tilde{x}] \leq_k [\tilde{y}]$ in $X'/\mathcal{R}$, and $x \leq y$.
follows since \( \varphi_2 \) is an order embedding. Therefore, \( X \) satisfies Condition \( O \).

**THEOREM 2.4.** For every c.o.s. \( X \) satisfying Conditions \( C \) and \( O \), \((X'/\mathcal{R}), \varphi_2\) is a \( T_1 \)-ordered c.o.s. compactification of \( X \). Furthermore, for any compact, regular, \( T_2 \)-ordered c.o.s. \( Y \) and for any continuous, increasing map \( f : X \to Y \), there is a unique, continuous, increasing extension \( f_\mathcal{R} : X'/\mathcal{R} \to Y \).

**PROOF.** The first assertion is an immediate corollary of Theorem 2.3. The second follows easily with the help of Proposition 1.3.

For any c.o.s. \( X \), let \( \omega_oX \) be the t.o.s. consisting of the poset \((X, \leq)\) with the weak topology induced by \( C'I'(X) \). Note that \( C'I'(X) \).

**PROPOSITION 2.5.** Let \( X \) be a c.o.s. satisfying Condition \( G \). Let \( \omega_oX \) be the identity map. Then \( \iota \) is an order isomorphism and a homeomorphism relative to ultrafilter convergence.

**PROOF.** It is obvious that \( \iota \) is a continuous order isomorphism. Let \( F \to x \) in \( \omega_oX \), where \( F \) is an ultrafilter. By Proposition 1.2, \( F \) is a maximal convex filter and \( f(F) \to f(x) \) implies \( f(\hat{F}) \to f(x) \) in \([0,1]\), for all \( f \in C'I'(X) \). Condition \( C \) thus guarantees that \( \hat{F} \to x \) in \( X \), and hence \( F \to x \) in \( X \).

**PROPOSITION 2.6.** If \( X \) is a c.o.s. satisfying Conditions \( G \) and \( O \), then \( \omega_oX \) is a \( T_{3,5} \)-ordered t.o.s.

**PROOF.** First observe that \( \omega_oX \) also satisfies Condition \( C \) and \( O \); \( O \) is obvious, and \( C \) follows from Proposition 2.5, since \( X \) and \( \omega_oX \) have the same ultrafilter convergence and hence, by Proposition 1.2, the same convergence of maximal convex filters.

For \( f \in C'I'(\omega_oX) \), let \( I \) be the closed interval \([0,1]\) indexed by \( f \), and let \( P = \Pi\{I_f : f \in C'I'(X)\} \) be equipped with the usual product order and product topology. Then \( P \) is a compact, \( T_\mathcal{F} \)-ordered t.o.s. Define \( \varphi_o : \omega_oX \to P \) by \( \varphi_o(x) = \hat{z} \), where \( \hat{z} : C'I'(\omega_oX) \to [0,1] \) is given by \( \hat{z}(f) = \hat{f}(z) \), for all \( f \in C'I'(\omega_oX) \). Since \( \omega_oX \) has the weak topology induced by \( C'I'(\omega_oX) = C'I'(X) \), and \( C'I'(\omega_oX) \) separates points in \( \omega_oX \) by Condition \( O \), \( \varphi_o \) is a topological embedding (see 8.12, [10]). By Condition \( O \), \( \varphi_o \) is also an order embedding.

Given a c.o.s. \( X \) satisfying \( C \) and \( O \), we introduce some additional functional notation. Let \( e_o \) be the evaluation embedding of the \( T_{3,5} \)-ordered t.o.s. \( \omega_oX \) into its Nachbin compactification \( \beta_oX \), and let \( e = e_o \cdot \iota : X \to \beta_o(\omega_oX) \). The unique extension of \( e \) to \( X' \) (guaranteed by Proposition 1.3) is denoted by \( e_* \), and the extension of \( e \) to \( X'/\mathcal{R} \) (guaranteed by Theorem 2.4) is denoted by \( e_\mathcal{R} \). If \( f \in C'I'(X) = C'I'(\omega_oX) \), the unique extensions of \( f \) in \( C'I'(X') \) and \( C'I'(\beta_o(\omega_oX)) \) (see Proposition 1.3 and 2.4) are denoted by \( f_* \) and \( f' \), respectively. The following commutative diagram is helpful in keeping track of these various maps.

\[
\begin{array}{ccc}
X & \xrightarrow{\iota} & X' \xrightarrow{\varphi} X'/\mathcal{R} \\
\downarrow e & & \downarrow e_* & \downarrow e_\mathcal{R} \\
\omega_oX & \to & \beta_o(\omega_oX) \\
\end{array}
\]

**THEOREM 2.7.** If \( X \) is any c.o.s. satisfying \( C \) and \( O \), then \( e_\mathcal{R} \) is an order isomorphism and a homeomorphism relative to ultrafilter convergence.

**PROOF.** Since \( |F| = |G| \) in \( X'/\mathcal{R} \) iff \( e_*(F) = e_*(G) \) iff \( e_\mathcal{R}(|F|) = e_\mathcal{R}(|G|) \), it follows that \( e_\mathcal{R} \) is one-to-one. Furthermore, \( e(X) \) is dense in \( \beta_o(\omega_oX) \), which implies that the extension \( e_\mathcal{R} \) is onto \( \beta_o(\omega_oX) \). It follows from Theorem 2.4 that \( e_\mathcal{R} \) is continuous and increasing. Finally, if \( \mathcal{U} \) is an ultrafilter on \( \beta_o(\omega_oX) \) and \( \mathcal{U} \to a \) in \( \beta_o(\omega_oX) \), then there is \( a \in X'/\mathcal{R} \) such that \( e_\mathcal{R}(\mathcal{U}) \to a \) in
Since the latter space is compact. It follows by uniqueness of filter limits in both spaces and
the continuity of \( e_R \) that \( e_R^{-1}(a) = a \).

If \( X \) is any convergence space, let \( \lambda X \) denote its topological modification (i.e., \( \lambda X \) is the set \(|X|\) equipped with the finest topological structure coarser than \( X \).) If \( X \) is a c.o.s. satisfying \( C \) and \( O \), we obtain from Proposition 2.5 and Theorem 2.7 that \( \lambda X = \omega \omega X \) and \( \lambda(X^*/\mathcal{R}) \) is a compact,
\( T_2 \)-ordered t.o.s. homeomorphic and order isomorphic under \( e_R \) to \( \beta_\omega(\omega_X) \). Let \( \varphi_\omega : \omega_X \to X^*/\mathcal{R} \) be defined by \( \varphi_\omega = \sigma \circ \varphi \circ i^{-1} = e_R \circ i^{-1} \).

**COROLLARY 2.8.** If \( X \) is a c.o.s. satisfying \( C \) and \( O \), then \( (\lambda(X^*/\mathcal{R}), \varphi_\omega) \) is the Nachbin
compactification of \( \omega_X = \lambda X \). If \( X \) is a \( T_{3.5} \)-ordered t.o.s., then \( (\lambda(X^*/\mathcal{R}), \varphi_\omega) \) is the Nachbin
compactification of \( X \).

One question which deserves clarification is the status of \( X^*/\mathcal{R} \) as a "quotient" of \( X^* \). We have
indeed equipped \( X^*/\mathcal{R} \) with the quotient convergence structure, but can we interpret
\( \leq_R \) as the "quotient order" relative to the order \( \leq \) defined on \( X^* \)? Various notions of "quotient order" have
been considered (for instance, see [5] and [8]), but the order \( \leq_R \) is generally different than these.
Instead of regarding the order and convergence structures of \( X^*/\mathcal{R} \) separately, we think that it is
appropriate to consider the notion of a "quotient c.o.s.", where order and convergence structures
are considered together. From this perspective, the next theorem indicates that \( X^*/\mathcal{R} \) is indeed a
quotient c.o.s. of \( X^* \), at least in the category of c.o.s.'s which satisfy Conditions \( C \) and \( O \).

**THEOREM 2.10.** For a c.o.s. \( X \), let \( X^* \) and \( X^*/\mathcal{R} \) be defined as before. Let \( Y \) be any c.o.s.
satisfying \( C \) and \( O \), and let \( h : X^*/\mathcal{R} \to Y \). Then \( h \) is continuous and increasing iff \( h \circ \sigma : X^* \to Y \)
is continuous and increasing.

**PROOF.** If \( h \) is continuous and increasing, the same is obviously true for \( h \circ \sigma \).

Conversely, suppose \( h \circ \sigma \) is continuous and increasing. Let \( \Phi \to [\mathcal{F}] \) in \( X^*/\mathcal{R} \); then there is
\( \mathcal{F}' \in [\mathcal{F}] \) and a filter \( \mathcal{A} \) on \( X^* \) such that \( \mathcal{A} \to \mathcal{F}' \to X^* \) and \( \Phi \geq \sigma(\mathcal{A}) \). Hence \( h \circ \sigma(\mathcal{A}) \to h \circ \sigma(\mathcal{F}') \)
in \( Y \), by continuity of \( h \circ \sigma \). But \( \Phi \geq \sigma(\mathcal{A}) \) and \( \sigma(\mathcal{F}') = [\mathcal{F}] \), so \( h(\Phi) \to h([\mathcal{F}]) \), implying that \( h \) is continuous.

To show that \( h \) is increasing, let \( e_Y \) be the natural map from \( Y \) into \( \beta_\omega(\omega_Y) \) and consider
\( g = e_Y \circ h \circ \sigma \circ \varphi : X \to \beta_\omega(\omega_Y) \). Since \( g : \omega_X \to \beta_\omega(\omega_Y) \) is also continuous and increasing,
there is a continuous, increasing extension \( g^* : \beta_\omega(\omega_X) \to \beta_\omega(\omega_Y) \) which makes the diagram below commute.

\[ X \xrightarrow{\varphi} X^* \xrightarrow{\sigma} X^*/\mathcal{R} \xrightarrow{h} Y \]

\[ e_R \downarrow \]

\[ \beta_\omega(\omega_X) \to \beta_\omega(\omega_Y) \]

\[ g^* \]

Thus \( e_Y \circ h \circ \sigma \circ \varphi = g^* \circ e_R \circ \sigma \circ \varphi \), and since \( \sigma \circ \varphi : X \to X^*/\mathcal{R} \) is a dense injection, \( e_Y \circ h = g^* \circ e_R \).
But \( e_Y \) is an order embedding, so \( h = e_Y^{-1} \circ g^* \circ e_R \), and \( h \) is increasing.

3. \( T_{3.5} \)-ORDERED CONVERGENCE ORDERED SPACES.

In this brief concluding section, we introduce the notion of a \( T_{3.5} \)-ordered c.o.s., describe
the largest regular, \( T_2 \)-ordered c.o.s. compactification of such a space, and interpret this compactification
in the language of category theory. The necessary categorical terminology can be found in
[7].

In [3], a convergence space \( X \) is defined to be completely regular if it allows a symmetric com-
In [9], it is shown that the Hausdorff, completely regular convergence spaces, which we shall refer to as $T_{3.5}$ convergence spaces, are precisely those convergence spaces which allow a regular, Hausdorff convergence space compactification.

Given a convergence space $X$, let $rX$ denote the regular modification of $X$ (i.e., $rX$ is the set $|X|$ equipped with the finest regular convergence structure coarser than the original convergence structure on $X$).

We define a c.o.s. $X$ which is regular and satisfies conditions $C$ and $O$ to be a $T_{3.5}$-ordered c.o.s. It follows by Proposition 2.5 that a $T_{3.5}$-ordered c.o.s. $X$ has the same ultrafilter convergence as its topological modification $\lambda X = \omega_2 X$.

THEOREM 3.1. Let $X$ be a $T_{3.5}$-ordered c.o.s. and let $\eta_sX = r(X^*/\mathcal{R})$ be the regular modification of $X^*/\mathcal{R}$. Then $(\eta_sX, \varphi_\mathcal{R})$ is a regular, $T_3$-ordered c.o.s. compactification of $X$. If $Y$ is a regular, $T_3$-ordered, compact c.o.s. and $f : X \rightarrow Y$ is continuous and increasing, then $f$ has a unique, continuous, increasing extension $f_* : \eta_sX \rightarrow Y$.

PROOF. By Theorem 2.3, $\varphi_\mathcal{R} : X \rightarrow X^*/\mathcal{R}$ is an order embedding and a homeomorphic embedding. By the functorial properties of the regular modification and the fact that $rX = X$, it follows that $\varphi_\mathcal{R} : X \rightarrow \eta_sX$ is continuous. Because $X^*/\mathcal{R}$ and $\eta_sX$ have the same ultrafilter convergence, it is easy to verify that the regular modification of $\varphi_\mathcal{R}(X)$ (considered as a subspace of $X^*/\mathcal{R}$) coincides with $\varphi_\mathcal{R}(X)$ considered as a subspace of $\eta_sX$. From this we see that $\varphi_\mathcal{R}^{-1}$ is also continuous, and the first assertion is established. The second assertion is an immediate consequence of Theorem 2.4.

We denote by $C$ the category of all $T_{3.5}$ ordered c.o.s.'s, with increasing continuous maps as morphisms; let $D$ be the full subcategory of $C$ consisting of all regular, compact, $T_3$-ordered c.o.s.'s. If $i : D \rightarrow C$ is the inclusion functor, it follows by Theorem 3.1 that the functor $\eta_s : C \rightarrow D$, which assigns to each object $X$ in $C$ its compactification $\eta_sX$ and to each morphism $f : X \rightarrow Y$ in $C$ the extension $f_* : \eta_sX \rightarrow \eta_sY$ whose existence follows by Theorem 3.1, is the left adjoint of $i$.

THEOREM 3.2. If $C$ and $D$ are the categories defined in the preceding paragraph, then $D$ is an epireflective subcategory of $C$.

If $X$ is a $T_{3.5}$-ordered t.o.s., it is generally not true that $\beta_sX = \eta_sX$, although it is true in this case that $\beta_sX = \lambda(\eta_sX)$.

The $T_{3.5}$ convergence spaces mentioned earlier in this section are the $T_{3.5}$-ordered c.o.s.'s for which the partial order is equality. Indeed, any $T_{3.5}$ convergence space $X$, equipped with the trivial order (equality), satisfies Condition $C$ and $O$ relative to $CI(X) = C^*(X)$, the set of all continuous maps from $X$ into $[0,1]$. For such a space $X$, $\eta_sX$ (which also has the trivial order) coincides with the largest regular, Hausdorff convergence space compactification of $X$ constructed in [9].

REFERENCES


