ABSTRACT. In this paper the author characterizes all those spaces $X$, for which $K_n(X, c_0)$ is proximinal in $L(X, c_0)$. Some examples were found that satisfy this characterization.

KEY WORDS AND PHRASES. Proximinal, best approximation, selection, extremal subspaces, n-width.

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1. INTRODUCTION.

The closed subset $A$ of the normed linear space $X$, is said to be “proximinal” in $X$ if for each $x \in X$, there is an element $y_x \in A$, such that:

$$d(x, A) = \inf\{\| x - y \| : y \in A \} = \| x - y_x \|,$$

where $d(x, A)$ is the distance of $x$ from $A$. The element $y_x$ is called a “best approximation” of $x$ in $A$. The best approximation need not be unique, and the set-valued function $P_A : X \to 2^A$ defined by

$$P_A = \{(y \in A ; d(x, A) = \| x - y \| \}$$

is called the metric projection of $X$ into $A$. If $A$ is proximinal in $X$ then $P_A(x) \neq \emptyset$ for each $x \in X$, in this case any function $f : X \to A$ satisfying that $f(x) \in P_A(x)$ for each $x \in X$, is called a “selection” for the metric projection $P_A$.

If $A$ is a subset of $X$, and $N$ is a subspace of $X$, then the “deviation” of $A$ from $N$ is defined to be

$$(A, N) = \sup\{(d(x, N); z \in A),$$

and the n-width of $A$ in $X$ is defined to be

$$d_n(A, X) = \inf\{(d(A, N); N \text{ is an n-dimensional subspace of } X).$$

If there is an n-dimensional subspace $N_0$ of $X$, such that $d_n(A, X) = d(A, N_0)$ then $d_n(A, X)$ is said to be “attained”, and the subspace $N_0$ is said to be an “extremal subspace” for $d_n(A, X)$. It is well known (see Garkavi [4]), that if $X^*$ is the dual space of the normed linear space $X$, then $d_n(A, X^*)$ is attained.

If $X$ and $Y$ are two normed linear spaces, then $L(X, Y)$ denotes the set of all bounded linear operators from $X$ to $Y$, $K(X, Y)$ the set of all compact operators in $L(X, Y)$, and $K_n(X, Y)$ the subset of $K(X, Y)$ consisting of all operators of rank $n$.

The proximinality of $K(X, Y)$ in $L(X, Y)$ were studied by several authors, (see for examples Feder [3], Lau [8], Mach [9], Mach and Ward [10], and Saatkamp [11]). Duetsch, Mach, and
Saatkamp [1], Kamal ([5], [6], and [7]) studied the proximinality of $K_n(X,Y)$ in $L(X,Y)$ and $K(X,Y)$ in details, however, one of the problems left unsolved is the problem 5.2.2 mentioned by Duetsch. Mach and Saatkamp [1], concerning the proximinality of $K_n(X,c_0)$, in $L(X,c_0)$ where $c_0$ is the space of all real sequences that converges to zero. The problem is divided into two parts, the first part is to characterize all those spaces $X$ for which $K_n(X,c_0)$ is proximinal in $L(X,c_0)$, and the second part is to show whether $K_n(X,c_0)$ is proximinal in $L(X,c_0)$ or not, when $X = c$ or $1_\infty$. Kamal [7] showed that $K_n(c,c_0)$ is not proximinal in $L(c,c_0)$, given a partial solution for the second part of the mentioned problem. Deutsch, Mach, and Saatkamp [1] showed that if $X = c_0$ or if $X^*$ is uniformly convex, then $K_n(X,c_0)$ is proximinal in $L(X,c_0)$, Kamal [6] showed that $K_n(1,c_0)$ is not proximinal in $L(1,c_0)$, also Kamal [7] showed that if $Q$ is a compact Hausdorff space that contains an infinite convergent sequence, then $K_n(C(Q),c_0)$ is not proximinal in $L(C(Q),c_0)$. In this paper a theorem is proved to characterize all those spaces $X$, for which $K_n(X,c_0)$ is proximinal in $L(X,c_0)$, this characterization includes $X = c_0$, $X$ for which $X^*$ is uniformly convex, and $X$ such that the metric projection from any of its $n$-dimensional subspaces, has a selection which is $\omega^*$-continuous at zero. A point worth mentioning is that although $c_0$ is a one codimensional subspace of $c$, there are spaces $X$ for which $K_n(X,c_0)$ is proximinal in $L(X,c_0)$, meanwhile $K_n(X,c)$ is not proximinal in $L(X,c)$, for example Deutsch, Mach and Saatkamp [1] showed that $K_n(c_0,c_0)$ is proximinal in $L(c_0,c_0)$, meanwhile Kamal [7] showed that $K_n(c_0,c)$ is not proximinal in $L(c_0,c)$.

The rest of introduction will cover some definitions, and known results that will be used later in Section 2.

If $X$ is a normed linear space then $c_0(X^*,\omega^*)$ denotes the Banach space of all bounded sequences $\{x_i\}$ in $X^*$ that converge to zero in the $\omega^*$-topology induced on $X^*$ by $X$, $c_0$ ($X^*$) is the Banach space of all sequences $\{x_i\}$ in $X^*$ that converge to zero in the topology defined on $X^*$ by its norm, and if $n \geq 1$ is any positive integer, then $c_0(X^*,n)$ denotes the union of all $c_0(N)$, where $N$ is an $n$-dimensional subspace of $X^*$. The norm on $c_0(X^*,\omega^*)$ is the suprimum norm. If $\{x_i\}$ is an element in $c_0(X^*,\omega^*)$ then for any positive integer $n \geq 1$, define

$$a_n(\{x_i\}) = \inf\{\|\{y_i\} - \{x_i\}\| : \{y_i\} \in c_0(X^*,n)\}$$

The following theorem can be obtained as a corollary, from the theorem of Dunford and Schwartz [2, p. 490].

**THEOREM 1.1.** Let $X$ be normed linear space. The mapping $T: L(X,c_0) \to c_0(X^*,\omega^*)$ defined by $T(z_i^*) = z_i$ where $i = 1, 2, \ldots$, and $z \in X$, is an isometric isomorphism. Furthermore $\alpha(K_n(X,c_0)) = c_0(X^*)$ and $\alpha(K_n(X,c_0)) = c_0(X^*,n)$.

As corollary of the Theorem 1.1, one can obtain the following:

**COROLLARY 1.2.** If $X$ is a normed linear space then for any positive integer $n \geq 1$, the set $K_n(X,c_0)$ is proximinal in $L(X,c_0)$ (resp. $K_n(X,c_0)$) if and only if $c_0(X^*,n)$ is proximinal in $c_0(X^*,\omega^*)$ (resp. $c_0(X^*)$).

According to Corollary 1.2 to study the proximinality of $K_n(X,c_0)$ in $L(X,c_0)$ (resp. $K(X,c_0)$), it is enough to study the proximinality of $c_0(X^*,n)$ in $c_0(X^*,\omega^*)$ (resp. $c_0(X^*)$).

2. **THE PROXIMALITY OF K_n(X,c_0) IN L(X,c_0).**

In this paper if $\{x_i\}$ is an element in $c_0(X^*,\omega^*)$, then $d_n(\{x_i\}, X^*)$ (resp. $\delta(\{x_i\}, N)$ denotes the $n$-width (resp. the deviation from $N$) of the subset $\{x_1, x_2, x_3, \ldots\}$ of $X^*$. $X^*$.

**THEOREM 2.1** Let $X$ be a normed linear space, and let $n \geq 1$ be any positive integer. If $\{x_i\}$ is a bounded sequence in $X^*$ then

$$a_n(\{x_i\}) = \max\{d_n(\{x_i\}, X^*), \lim \|x_i\|\}.$$ 

Furthermore there is an $n$-dimensional subspace $N_0$ of $X^*$, such that $a_n(\{x_i\}) = d(\{x_i\}, c_0(N))$. 

$a_n(\{x_i\}) = \max\{d_n(\{x_i\}, X^*), \lim \|x_i\|\}$. 

Furthermore there is an $n$-dimensional subspace $N_0$ of $X^*$, such that $a_n(\{x_i\}) = d(\{x_i\}, c_0(N))$. 


PROOF. First it will be shown that \( a_n({z_i}) \geq \max\{d_n({x_i}, N^*), \|z_i\|\} \). By Garkavi [4], there is an \( n \)-dimensional subspace \( N_o \) of \( X^* \) such that \( \delta({x_i}, N) = d_n({x_i}, X^*) \). For each \( i = 1, 2, \ldots \), let \( z_i \) be a best approximation for \( x_i \) from \( N_o \), and let \( \varepsilon > 0 \) be given. Define the sequence \( \{y_i\} \) in \( c_o(N_o) \) as follows.

\[
y_i = \begin{cases} z_i & \text{if } i \leq m \\ 0 & \text{if } i > m. \end{cases}
\]

Then

\[
a_n({z_i}) \leq \|x_i - y_i\| = \sup\{\|x_i - y_i\| : i = 1, 2, \ldots\} = \max\{\max\{\|x_i - y_i\| : i = 1, 2, \ldots, m\}, \sup\{\|x_i\| : i = m + 1, m + 2, \ldots\}\}
\]

\[
\leq \max\{d_n({z_i}, X^*), \|z_i\| + \varepsilon\}.
\]

Since \( \varepsilon \) is arbitrary it follows that \( a_n({z_i}) \leq \max\{d_n({z_i}, X^*), \|z_i\|\} \). Second to show that \( a_n({z_i}) \geq \max\{d_n({z_i}, X^*), \|z_i\|\} \), one should notice first that \( a_n({z_i}) \geq \|z_i\| \), indeed if \( \{y_i\} \in c_o(X^*, n) \) then

\[
\|x_i - y_i\| = \sup\{\|x_i - y_i\| \geq \|x_i - y_i\| = \|z_i\| \}
\]

Let \( \varepsilon > 0 \) be given, there is an \( n \)-dimensional subspace \( N^* \) of \( X^* \), and a sequence \( \{y_i\} \in c_o(N^*) \) such that \( a_n({z_i}) \geq \|x_i - y_i\| - \varepsilon \). Therefore

\[
\|x_i - y_i\| = \sup\{\|x_i - y_i\| \geq \sup\{\|x_i - y_i\| \geq \sup\{\|x_i - y_i\| \} \}
\]

\[
\geq \|x_i - y_i\| = \|z_i\| + \varepsilon
\]

\[
\leq \max\{d_n({z_i}, X^*), \|z_i\| \} + \varepsilon = a_n({z_i}) + \varepsilon.
\]

But \( \varepsilon \) is arbitrary so \( d({z_i}, c_o(N^*)) = a_n({z_i}) \).

THEOREM 2.2. Let \( X \) be a normed linear space. For any positive integer \( n \geq 1 \), \( K_n(X, c_o) \) is proximinal in \( K(X, c_o) \).

PROOF. Let \( \{x_i\} \) be an element in \( c_o(X^*) \), by Corollary 1.2, it is enough to find an element \( \{y_i\} \in c_o(X^*, n) \) such that \( \|x_i - y_i\| = a_n({z_i}) \). Since \( \lim_{i \to \infty} \|z_i\| = 0 \) it follows that \( \sup_{i \to \infty} \|z_i\| = 0 \), thus by Theorem 2.1, \( a_n({z_i}) = d_n({z_i}, X^*) \). Let \( N_o \) be an extremal subspace for \( d_n({z_i}, X^*) \), and for each \( i = 1, 2, \ldots \), let \( y_i \) be a best approximation for \( x_i \) from \( N_o \). Since \( \lim_{i \to \infty} \|x_i\| = 0 \), it follows that

\[
\|x_i - y_i\| = \sup\{\|x_i - y_i\| = \|z_i\| = a_n({z_i})\}.
\]

LEMMA 2.3. Let \( X \) be a normed linear space, and let \( \{x_i\} \) be a bounded sequence in \( X^* \).

a) If \( d_n({z_i}, X^*) > \sup_{i \to \infty} \|z_i\| \) then \( a_n({z_i}) \) is attained.
b) If \( d_n(\{x_i\}, X^*) \leq \lim d(x_i, N_o) \), and there is an extremal subspace \( N_o \) for \( d_n(\{x_i\}, X^*) \) such that
\[
\lim d(x_i, N_o) < \lim \|x_i\|, \quad \text{then } a_n(\{x_i\}) \text{ is attained.}
\]

PROOF. a) Assume that \( N \) is an extremal subspace for \( d_n(\{x_i\}, X^*) \), and let
\[
a = d_n(\{x_i\}, X^*) - \lim \|x_i\|, \quad \text{then there is a positive integer } m \geq 1 \text{ such that for each } i \geq m, \text{ one has}
\]
\[
\|x_i\| \leq \lim \|x_i\| + a. \quad \text{For each } i \leq m, \text{ let } z_i \text{ be a best approximation for } x_i \text{ from } N_o, \text{ and define the sequence } \{y_i\} \text{ in } c_0(N_o) \text{ as follows.}
\]
\[
y = \begin{cases} 
  z_i & \text{if } i \leq m \\
  0 & \text{if } i > m.
\end{cases}
\]
Then
\[
\|\{x_i\} - \{y_i\}\| = \max\{\max\{\|x_i - z_i\| ; i = 1, 2, \ldots, m\}, \sup\{\|x_i\| ; i = m + 1, m + 2, \ldots\}\}
\leq \max\{d(\{x_i\}, N_o), \lim \|x_i\| + a\}
\leq d_n(\{x_i\}, X^*) = a_n(\{x_i\}).
\]

b) Let \( \epsilon_i \) be a sequence of positive real numbers, satisfying that \( \lim_{i \to \infty} \epsilon_i = 0 \), for each
\[
i = 1, 2, \ldots, d(x_i, N_o) \leq \beta + \epsilon_i \text{ and for each } i = 1, 2, \ldots, \|x_i\| \leq \alpha + \epsilon_i. \quad \text{For each } i = 1, 2, \ldots, \text{ let } z_i \text{ be a best approximation for } x_i \text{ from } N_o, \text{ and define the sequence } \{y_i\} \text{ in } N_o \text{ as follows.}
\]
\[
y_i = \begin{cases} 
  y_{i-1} & \text{if } \epsilon_i \geq \gamma \\
  z_i & \text{if } \epsilon_i < \gamma.
\end{cases}
\]
Since \( \{x_i\} \) is a bounded sequence in \( N_o \), and \( \lim_{i \to \infty} \epsilon_i = 0 \), it follows that \( \{y_i\} \in c_0(N_o) \).
Furthermore for each \( i = 1, 2, \ldots, \) if \( \epsilon_i > \gamma \) then
\[
\|x_i - y_i\| \leq d(x_i, N_o) \leq d_n(\{x_i\}, X^*) \leq a(\{x_i\}),
\]
and if \( \epsilon_i < \gamma \) then
\[
\|x_i - y_i\| \leq (1 - \frac{\epsilon_i}{\gamma}) \|x_i\| + \frac{\epsilon_i}{\gamma} \|x_i - z_i\|
\leq (1 - \frac{\epsilon_i}{\gamma})(\alpha + \epsilon_i) + \frac{\epsilon_i}{\gamma}(\alpha - \gamma)
\leq \alpha = a_n(\{x_i\}).
\]
Thus \( \|\{x_i\} - \{y_i\}\| = a_n(\{x_i\}) \).

Lemma 2.4 is a continuation for Lemma 2.3.

LEMMA 2.4. Let \( X \) be a normed space, and let \( \{x_i\} \) be a bounded sequence in \( X^* \). Assume that \( d_n(\{x_i\}, X^*) = \lim d(x_i, N) \), and for each extremal subspace \( N \) for \( d_n(\{x_i\}, X^*) \) one has
\[
\lim d(x_i, N) = \lim \|x_i\| = \alpha. \quad \text{Let } N \text{ be a extremal subspace for } d_n(\{x_i\}, X^*), \text{ and for each } i = 1, 2, \ldots, \text{ define}
\]
\[
\epsilon_i = \begin{cases} 
  0 & \text{if } \|x_i\| \leq \alpha \\
  \|x_i\| - \alpha & \text{if } \|x_i\| > \alpha.
\end{cases}
\]
\[
\delta_i = \alpha - d(x_i, N_o), \quad \text{and } \alpha_i = \begin{cases} 
  \epsilon_i & \text{if } \epsilon_i + \delta_i = 0 \\
  \epsilon_i + \delta_i & \text{if } \epsilon_i + \delta_i \neq 0.
\end{cases}
\]
If \( \lim_{i \to \infty} \alpha_i = 0 \) then \( a_n(\{x_i\}) \) is attained.

PROOF. Let \( z_i \) be a best approximation for \( x_i \) from \( N_o \), and let \( y_i = \alpha_i \cdot z_i \), then the sequence \( \{y_i\} \) is an element in \( c_0(N_o) \). Furthermore for each \( i = 1, 2, \ldots, \)
\[
\|x_i - y_i\| \leq (1 - \alpha_i) \|x_i\| + \alpha_i \|x_i - z_i\|
\leq (1 - \alpha_i)(\alpha + \epsilon_i) + \alpha_i(\alpha - \delta_i)
\leq \alpha + \epsilon_i - \alpha_i(\epsilon_i + \delta_i).
\]
If \( \alpha_i = 0 \) then \( \epsilon_i = 0 \) so \( ||x_i - y_i|| = \alpha \), and if \( \alpha_i \neq 0 \) then
\[
||x_i - y_i|| \leq \alpha + \epsilon_i - \frac{\epsilon_i}{\epsilon_i + \delta_i} (\epsilon_i + \delta_i) = \alpha.
\]

**Definition 2.5.** Let \( X \) be a normed linear space. The bounded sequence \( \{x_i\} \) in \( c_0(X^*, \omega^*) \) is said to be an “n-border” sequence if it satisfies the following,

1. \( \lim_{i \to \infty} ||x_i|| \) exists, and for each extremal subspace \( N \) for \( d_n(\{x_i\}, X^*) \), one has
\[
\lim_{i \to \infty} d(x_i, N) = \lim_{i \to \infty} ||x_i|| = d_n(\{x_i\}, X^*).
\]
2. For each extremal subspace \( N \) for \( d_n(\{x_i\}, X^*) \) if \( \epsilon_i, \delta_i \) and \( \alpha_i \) as in Lemma 2.4 then \( \lim_{i \to \infty} \alpha_i > 0 \).

**Theorem 2.6.** Let \( X \) be a normed linear space, and let \( n \geq 1 \) be a positive integer. Then \( K_n(X, c_0) \) is proximinal in \( L(X, c_0) \) if and only if for each n-border sequence \( \{x_i\} \) in \( X^* \), \( a_n(\{x_i\}) \) is attained.

**Proof.** If there is an n-border sequence \( \{x_i\} \) in \( X^* \) such that \( a_n(\{x_i\}) \) is not attained, then since \( \{x_i\} \in c_0(X^*, \omega^*) \), it follows by Corollary 1.2 that \( K_n(X, c_0) \) is not proximinal in \( L(X, c_0) \). To prove the other part, let \( \{x_i\} \) be an element in \( c_0(X^*, \omega^*) \). If \( d_n(\{x_i\}, X^*) > \lim_{i \to \infty} ||x_i|| \), or if \( d_n(\{x_i\}, X^*) \leq \lim_{i \to \infty} ||x_i|| \) and there is an extremal subspace \( N \) for \( d_n(\{x_i\}, X^*) \), such that
\[
\lim_{i \to \infty} d(x_i, N) < \lim_{i \to \infty} ||x_i|| \text{ then by Lemma 2.3, } a_n(\{x_i\}) \text{ is attained.}
\]
Assume that \( d_n(\{x_i\}, X^*) = \lim_{i \to \infty} ||x_i|| \), and for each extremal subspace \( N \) for \( d_n(\{x_i\}, X^*) \), one has
\[
\lim_{i \to \infty} d(x_i, N) = \lim_{i \to \infty} ||x_i|| \text{ then by Lemma 2.3, } a_n(\{x_i\}) \text{ is attained.}
\]
Assume that \( d_n(\{x_i\}, X^*) = \lim_{i \to \infty} ||x_i|| \), and for each extremal subspace \( N \) for \( d_n(\{x_i\}, X^*) \) one has \( \lim_{i \to \infty} \alpha_i > 0 \). Let \( \alpha = d_n(\{x_i\}, X^*) \) and let \( \{x_i\} \) be the largest subsequence of \( \{x_i\} \) satisfying that \( ||x_i|| > \alpha \) for each \( i_k \). Thus for each \( i \), if \( x_i \) is not an element in \( \{x_i\} \) then \( ||x_i|| \leq \alpha \). The sequence \( \{x_i\} \) is an n-border sequence in \( X^* \), and there is an n-dimensional subspace \( N \) of \( X^* \), and a sequence \( \{z_i\} \in c_0(N) \) such that
\[
||\{z_i\} - \{x_i\}|| = a_n(\{x_i\}) = \alpha.
\]
Define the sequence \( \{y_i\} \) in \( N \) as follows.
\[
y_i = \begin{cases} 
  x_i & \text{if } ||x_i|| > \alpha \\
  0 & \text{if } ||x_i|| \leq \alpha.
\end{cases}
\]
Then \( \{y_i\} \in c_0(N) \), and \( ||\{x_i\} - \{y_i\}|| = \alpha = a_n(\{x_i\}) \).

**Corollary 2.7.** Let \( X \) be a normed linear space, and let \( n \geq 1 \) be a positive integer. If \( X^* \) is uniformly convex then \( K_n(X, c_0) \) is proximinal in \( L(X, c_0) \).

**Proof.** Let \( \{x_i\} \) be an n-border sequence in \( X^* \), and let \( \alpha = \lim_{i \to \infty} ||x_i|| \). Without loss of generality assume that \( x_i \neq 0 \) for each \( i \). Let \( N \) be any extremal subspace for \( d_n(\{x_i\}, X^*) \), and let \( y_i \) be the best approximation for \( x_i \) from \( N \). Since
\[
\frac{||x_i||}{||x_i||} = 1, \quad \frac{||x_i - y_i||}{\alpha} \leq 1,
\]
and
\[
\lim_{i \to \infty} \frac{||x_i||}{||x_i||} = \lim_{i \to \infty} \frac{||x_i - y_i||}{\alpha} \leq 1,
\]
and
\[
\frac{||x_i - y_i||}{\alpha} = \lim_{i \to \infty} \left( \frac{\alpha + ||x_i||}{\alpha} \right) \frac{||x_i - y_i||}{\alpha} \leq \lim_{i \to \infty} \left( \frac{\alpha + ||x_i||}{\alpha} \left( \frac{||x_i||}{\alpha} \right) \right) \frac{||x_i - y_i||}{\alpha} = 2.
\]
It follows by the fact that \( X^* \) is uniformly convex that
\[
\lim_{i \to \infty} \frac{||x_i||}{||x_i||} - \frac{||x_i - y_i||}{\alpha} = 0.
\]
But then
\[
\lim_{i \to \infty} \frac{||x_i||}{||x_i||} - \frac{||x_i - y_i||}{\alpha} = 0.
\]
Corollary 2.7 was proved by Deutsch, Mach, and Saatkamp \([1]\) in a different way.

**Corollary 2.8.** Let \( X \) be a normed linear space, and let \( n \geq 1 \) be a positive integer. If for each n-dimensional subspace \( N \) of \( X^* \), the metric projection \( P_n \) has a selection which is \( \omega^* \)-continuous at zero, then \( K_n(X, c_0) \) is proximinal in \( L(X, c_0) \).

**Proof.** Let \( \{x_i\} \) be an element in \( c_0(X^*, \omega^*) \) and let \( N \) be an extremal subspace for \( d_n(\{x_i\}, X^*) \). Since the metric projection \( P_N \) has a selection which is \( \omega^* \)-continuous at zero, it follows that there is a sequence \( \{y_i\} \) in \( N \), satisfying that \( y_i \in P_N(x_i) \) for each \( i \), and that \( \{y_i\} \)
converges \( \omega^* \)-to zero. But \( N \) is of finite dimension, thus \( \{y_i\} \in c_0(N) \).

Furthermore

\[
\| x_i - y_i \| = \delta(x_i, N) = d_n(x_i, X^*) = a_n(x_i).
\]

From Corollary 2.8 one concludes that for each positive integer \( n \geq 1 \), if \( X = c_0 \) or \( l_p, l < p < \infty \), then \( K_n(X, c_0) \) is proximinal in \( L(X, c_0) \). Proposition 2.9 clarify that. The fact that \( K_n(c_0, c_0) \) is proximinal in \( L(c_0, c_0) \) was proved first by Deutsch, Mach, and Saatkamp [1].

**PROPOSITION 2.9.** Let \( n \geq 1 \) be a positive integer and let \( X = c_0 \) or \( l_p, l < p < \infty \). The metric projection \( P_N \) from \( X^* \) onto any of its \( n \)-dimensional subspace \( N \), has a selection which is \( \omega^* \)-continuous at zero.

**PROOF.** Let \( N \) be any \( n \)-dimensional subspace of \( X^* \), \( \{x_i\} \) be any bounded sequence in \( X^* \) that converges \( \omega^* \)-to zero, and let \( \{y_i\} \) by any sequence in \( N \), satisfying that \( y_i \in P_N(x_i) \) for each \( i \).

It will be shown that \( \{y_i\} \in c_0(N) \). The sequence \( \{y_i\} \) is a bounded sequence in a finite dimensional subspace of \( X^* \), so it has a convergent subsequence \( \{y_{i_k}\} \) that converges to \( y_0 \) in \( N \), it will be shown that \( y_0 = 0 \). Assume not, and without loss of generality assume that \( \{y_i\} \) converges to \( y_0 \), and that \( X^* = l_p, l < p < \infty \). Let \( t_i = x_i - (y_i - y_0) \), \( r_i = x_i - y_i \), and let \( \varepsilon > 0 \) be such that \( \varepsilon < \| y_0 \| \), then as in Proposition 3 of Mach [9], there is a positive integer \( m \geq 1 \) such that for each \( i \geq m \) one has

\[
\| t_i - y_0 \| \leq \| t_i \| + \| y_0 \| = \| x_i - y_i \| + \| y_0 \| \leq \| x_i - (y_i - y_0) \|.
\]

So for each \( i > m \) one has \( \| x_i - (y_i - y_0) \| > \| x_i - y_i \| \), which contradict the fact that \( \| x_i - y_i \| = d(x_i, N) \), therefore \( y_0 = 0 \).

**REFERENCES**