ON CERTAIN MEROMORPHIC FUNCTIONS WITH POSITIVE COEFFICIENTS

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ABSTRACT. In this paper, we introduce a new class $T_p(\alpha)$ of meromorphic functions with positive coefficients in $D = \{z : 0 < |z| < 1\}$. The aim of the present paper is to prove some properties for the class $T_p(\alpha)$.

KEY WORDS AND PHRASES. Meromorphic function, meromorphically starlike and convex.

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1. INTRODUCTION.

Let $A_\alpha$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n \quad (p = 1, 3, 5, \ldots)$$

which are analytic in $D = \{z : 0 < |z| < 1\}$ with a simple pole at the origin with residue one there.

A function $f(z) \in A_\alpha$ is said to be meromorphically starlike of order $\alpha$ if it satisfies

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha$$

for some $0 \leq \alpha < 1$ and for all $z \in D$.

Further, a function $f(z) \in A_\alpha$ is said to be meromorphically convex of order $\alpha$ if it satisfies

$$\Re \left\{ \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} > \alpha$$

for some $0 \leq \alpha < 1$ and for all $z \in D$.

Some subclasses of $A_1$, when $p = 1$ were recently introduced and studied by Pommerenke [1], Miller [2], Mogra, et al [3], and Cho, et al [4].

Let $T_p$ be the subclass of $A_\alpha$ consisting of functions

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n \quad (a_n \geq 0).$$

A function $f(z) \in T_p$ is said to be a member of the class $T_p(\alpha)$ if it satisfies

$$\frac{|z^{p+1}f(p)(z) + p!|}{|z^{p+1}f(p)(z) - p!|} < \alpha$$

for some $0 \leq \alpha < 1$ and for all $z \in D$.

In this paper we present a systematic study of the various properties of the class $T_p(\alpha)$ including distortion theorems and starlikeness and convexity properties.
2. DISTORTION THEOREMS.

We begin with the statement and the proof of the following coefficient inequality.

THEOREM 2.1. A function \( f(z) \in T_p \) is in the class \( T_p(\alpha) \) if and only if

\[
\sum_{n=p}^{\infty} \binom{n}{p} a_n \leq \frac{2\alpha}{1+\alpha},
\]

where

\[
\binom{n}{p} = \frac{n(n-1)\cdots(n-p+1)}{p!}.
\]

PROOF. Assuming that (2.1) holds for all admissible \( \alpha \), we have

\[
|z^p + 1/f^p(z) + p!| - \alpha |z^p + 1/f^p(z) - p!|
\]

\[
= \sum_{n=p}^{\infty} \frac{n!}{(n-p)!} a_n |z|^p + 1/f^p(z) + p! - \sum_{n=p}^{\infty} \frac{n!}{(n-p)!} a_n |z|^p + 1/f^p(z) - p! \]

\[
\leq \sum_{n=p}^{\infty} \frac{n!}{(n-p)!} (1 + \alpha) a_n |z|^{p+1} - 2\alpha \cdot p!.
\]

Therefore, letting \( z \to 1^{-} \), we obtain

\[
\sum_{n=p}^{\infty} \frac{n!}{(n-p)!} (1 + \alpha) a_n - 2\alpha \cdot p! \leq 0
\]

which shows that \( f(z) \in T_p(\alpha) \).

Conversely, if \( f(z) \in T_p(\alpha) \), then

\[
|z^p + 1/f^p(z)| = \frac{\sum_{n=p}^{\infty} \frac{n!}{(n-p)!} a_n |z|^p + 1/f^p(z) + p!}{2\cdot p! - \sum_{n=p}^{\infty} \frac{n!}{(n-p)!} a_n |z|^p + 1/f^p(z) - p!} \leq \alpha
\]

\([z \in D]\). (2.4)

Since \( \text{Re}(z) \leq |z| \) for all \( z \), (2.4) gives

\[
\text{Re} \left\{ \frac{\sum_{n=p}^{\infty} \frac{n!}{(n-p)!} a_n |z|^p + 1}{2\cdot p! - \sum_{n=p}^{\infty} \frac{n!}{(n-p)!} a_n |z|^p + 1} \right\} < \alpha \quad \text{[z \in D].} \]

Choose values of \( z \) on the real axis so that \( z^p + 1/f^p(z) \) is real. Upon clearing the denominator in (2.5) and letting \( z \to 1^{-} \), we have

\[
\sum_{n=p}^{\infty} \frac{n!}{(n-p)!} (1 + \alpha) a_n \leq 2\alpha \cdot p!
\]

which is equivalent to (2.1). Thus we complete the proof of Theorem 2.1.

Taking \( p = 1 \) in Theorem 1, we have

COROLLARY 2.1. \( f(z) \in T_1(\alpha) \) if and only if

\[
\sum_{n=1}^{\infty} n a_n \leq \frac{2\alpha}{1+\alpha}.
\]

THEOREM 2.2. If \( f(z) \in T_p(\alpha) \), then

\[
|f^{(j)}(z)| \geq \frac{j!}{|z|^{j+1}} \left( \frac{p^{2}\alpha}{(p-j)!(1+\alpha)} \right) |z|^{p-j}
\]

and

\[
|f^{(j)}(z)| \leq \frac{j!}{|z|^{j+1}} \left( \frac{p^{2}\alpha}{(p-j)!(1+\alpha)} \right) |z|^{p-j}
\]
for $z \in D$, where $0 \leq j \leq p$ and $0 < \alpha \leq \frac{j!(p-j)}{p!}$. Equalities in (2.8) and (2.9) are attained for the function

$$f(z) = \frac{1}{z} + \frac{2\alpha}{1+\alpha} z^p.$$  

PROOF. It follows from Theorem 2.1 that

$$\sum_{n=p}^{\infty} \frac{n!}{(n-j)!} a_n \leq \sum_{n=p}^{\infty} \frac{n!(1+\alpha)}{p!} a_n \leq 2\alpha.$$

Therefore, we have

$$|f^{(j)}(z)| \geq \frac{j!}{|z|^{j+1}} - \sum_{n=p}^{\infty} \frac{n!}{(n-j)!} |z|^{n-j} \geq \frac{j!}{|z|^{j+1}} - \frac{p!2\alpha}{(p-j)(1+\alpha)} |z|^{p-j}$$

and

$$|f^{(j)}(z)| \leq \frac{j!}{|z|^{j+1}} + \sum_{n=p}^{\infty} \frac{n!}{(n-j)!} |z|^{n-j} \leq \frac{j!}{|z|^{j+1}} + \frac{p!2\alpha}{(p-j)(1+\alpha)} |z|^{p-j}.$$

Taking $j = 0$ in Theorem 2.2, we have

COROLLARY 2.2 If $f(z) \in T_p(\alpha)$, then

$$\frac{1}{|z|} - \frac{2\alpha}{1+\alpha} |z|^p \leq |f(z)| \leq \frac{1}{|z|} + \frac{2\alpha}{1+\alpha} |z|^p$$

for $z \in D$. Equalities in (2.14) are attained for the function $f(z)$ given by (2.10).

Making $j = 1$ in Theorem 2.3, we have

COROLLARY 2.3. If $f(z) \in T_p(\alpha)$, then

$$\frac{1}{|z|} - \frac{2\alpha p}{1+\alpha} |z|^{p-1} \leq |f'(z)| \leq \frac{1}{|z|^2} + \frac{2\alpha |z|^{p-1}}{1+\alpha}$$

for $z \in D$, where $0 < \alpha \leq \frac{1}{2p-1}$. Equalities in (2.15) are attained for the function $f(z)$ given by (2.10).

Letting $p = 1$ in Theorem 2.2, we have

COROLLARY 2.4. If $f(z) \in T_1(\alpha)$, then

$$\frac{1}{|z|} - \frac{2\alpha}{1+\alpha} |z| \leq |f(z)| \leq \frac{1}{|z|} + \frac{2\alpha}{1+\alpha} |z|$$

and

$$\frac{1}{|z|^2} - \frac{2\alpha}{1+\alpha} \leq |f'(z)| \leq \frac{1}{|z|^2} + \frac{2\alpha}{1+\alpha}$$

for $z \in D$. Equalities in (2.16) and (2.17) are attained for the function

$$f(z) = \frac{1}{z} + \frac{2\alpha}{1+\alpha} z.$$

3. STARLIKE AND CONVEXITY.

THEOREM 3.1. If $f(z) \in T_p(\alpha)$, then $f(z)$ is meromorphically starlike of order $\delta$ ($0 \leq \delta < 1$) in $|z| < r_1$, where

$$r_1 = \inf \left\{ \left( \frac{n}{p} \right)^{\frac{1}{1+\delta}} \left( 1 - \frac{\alpha(n+2-\delta)}{2\alpha(n+2-\delta)} \right)^{\frac{1}{n+1}} \right\}.$$  

(3.1)

The result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{2\alpha}{p(1+\alpha)} z^n \quad (n \geq p).$$

PROOF. It is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} + 1 \right| \leq 1 - \delta$$

for $|z| < r_1$. We note that
\[
\left| \frac{zf'(z)}{f(z)} + 1 \right| = \left| \sum_{n=0}^{\infty} \frac{(n+1)a_n z^n}{1 + \sum_{n=0}^{\infty} p \alpha_n z^n} \right| \leq \sum_{n=0}^{\infty} \frac{|n+1| a_n z^n}{1 - \sum_{n=0}^{\infty} p \alpha_n z^n}.
\]

(3.4)

Therefore, if
\[
\sum_{n=0}^{\infty} \frac{n+2-\delta}{1-\delta} a_n |z|^{n+1} \leq 1,
\]

(3.5)

then (3.3) holds true. Further, using Theorem 2.1, it follows from (3.5) that (3.3) holds true if
\[
\frac{n+2-\delta}{1-\delta} |z|^{n+1} \leq \frac{(p)(1+\alpha)}{2\alpha} \quad (n \geq p),
\]

(3.6)

or
\[
|z| \leq \frac{\left(\frac{n}{p}(1+\alpha)(1-\delta)\right)^{n+1}}{2\alpha n(n+2-\delta)} \quad (n \geq p).
\]

(3.7)

This completes the proof of Theorem 3.1

**Theorem 3.2.** If \( f(z) \in T_p(\alpha) \), then \( f(z) \) is meromorphically convex of order \( \delta \) (0 \( \leq \delta \) < 1) in \( |z| < r_2 \), where
\[
r_2 = \inf_{n \geq p} \left\{ \left(\frac{n}{p}\right)(1+\alpha)(1-\delta)\right\}^{n+1} \quad (3.8)
\]

The result is sharp for the function \( f(z) \) given by (3.2).

**Proof.** Note that we have to prove that
\[
\left| \frac{zf''(z)}{f'(z)} + 2 \right| \leq 1 - \delta
\]

(3.9)

for \( |z| < r_2 \). Since
\[
\left| \frac{zf''(z)}{f'(z)} + 2 \right| = \left| \sum_{n=0}^{\infty} \frac{n(n+1)a_n z^{n-1}}{1 - \sum_{n=0}^{\infty} p \alpha_n z^n} \right| \leq \sum_{n=0}^{\infty} \frac{n(n+1)a_n |z|^{n+1}}{1 - \sum_{n=0}^{\infty} p \alpha_n |z|^{n+1}}.
\]

(3.10)

we see that if
\[
\sum_{n=0}^{\infty} \frac{n(n+2-\delta)}{1-\delta} a_n |z|^{n+1} \leq 1,
\]

(3.11)

Or
\[
\frac{n(n+2-\delta)}{1-\delta} |z|^{n+1} \leq \frac{(p)(1+\alpha)}{2\alpha} \quad (n \geq p),
\]

(3.12)

then (3.9) holds true. Therefore, \( f(z) \) is meromorphically convex of order \( \delta \) in \( |z| < r_2 \).

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