ON REGULAR AND SIGMA-SMOOTH TWO VALUED MEASURES AND LATTICE GENERATED TOPOLOGIES

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ABSTRACT. Let $X$ be an abstract set and $L$ a lattice of subsets of $X$. $I(L)$ denotes the non-trivial zero one valued finitely additive measures on $A(L)$, the algebra generated by $L$, and $IR(L)$ those elements of $I(L)$ that are $L$-regular. It is known that $I(L)=IR(L)$ if and only if $L$ is an algebra. We first give several new proofs of this fact and a number of characterizations of this in topological terms.

Next we consider, $I(\sigma^*, L)$ the elements of $I(L)$ that are $\sigma$-smooth on $L$, and $IR(\sigma, L)$ those elements of $I(\sigma^*, L)$ that are $L$-regular. We then obtain necessary and sufficient conditions for $I(\sigma^*, L)=IR(\sigma, L)$, and in particular, we obtain conditions in terms of topological demands on associated Wallman spaces of the lattice.

KEY WORDS AND PHRASES. Regular and sigma smooth two valued measures, normal lattices, regular lattices, $T_2$ lattices, countably paracompact and countably bounded, separation and semi-separation of lattices, pre-measures, $I$-lattice, etc.

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1. INTRODUCTION

In this paper we wish to determine when certain classes of measures are equal, and to obtain necessary and sufficient conditions for such equality to hold, emphasizing topological characterizations.

To be specific let $X$ be an abstract set, $L$ a lattice of subsets of $X$. Let $A(L)$ denote the algebra generated by the lattice $L$, and $I(L)$ the collection of non-trivial zero-one valued finitely additive measures on $A(L)$. $IR(L)$ will denote measures in $I(L)$ that are $L$-regular on $A(L)$, i.e. if $\mu \in IR(L)$ and $B \in A(L)$ then there exists a $L \in L$ such that $B \supseteq L$ and $\mu(B)=\mu(L)$. $I(\sigma^*, L)$ will denote those elements of $I(L)$ that are $\sigma$-smooth on $L$, i.e. if $L_n \in L$, $n=1, 2, \ldots \infty$ and $L_n \uparrow \emptyset$ then for $\mu \in I(\sigma^*, L)$, $\lim \mu(L_n)=0$. $IR(\sigma, L)$ will denote those measures in $I(\sigma^*, L)$ that are $L$-regular.
The first area of concern is when $I(L) = IR(L)$. It is well known that this is true iff $L$ is an algebra. We give several proofs of this, highlighting topological considerations, to be more precise:

1. **The lattice $V(L)$ (see below for definitions) in the space $I(L)$ is regular.**
2. **The topology of closed sets $\tau V(L)$ in $I(L)$ is $T_1$.**
3. **The lattice of sets $V(L)$ in $I(L)$ is disjunctive.**

The second main area of concern is determining conditions for $I(\sigma*,L) = IR(\sigma,L)$, and conversely what this implies for the lattice. We show (see below for definitions) that $I(\sigma*,L) = IR(\sigma,L)$ is equivalent to:

1. **The lattice $V(\sigma,L)$ in the space $I(\sigma*,L)$ is regular.**
2. **The lattice of sets $V(L)$ in $I(\sigma*,L)$ is disjunctive.**

2. **BACKGROUND AND NOTATION**

We begin by reviewing some notation and terminology which is fairly standard (see Alexandroff [1], Frolik [4], and Szeto [7]). We supply background material for the reader's convenience.

Let $X$ be an abstract set, and $L$ a lattice of subsets of $X$, where $\emptyset \in L$. A delta lattice is one that is closed under countable intersections, and the delta lattice generated by $L$ is denoted $\delta(L)$. In addition, $L$ is complement generated iff for every element $L \in L$, there exists a sequence of subsets $A_1 \in \delta(L), A_2 \in \delta(L), \ldots$, and $L = \cap A_i, i = 1, 2, \ldots$ (where $'$ denotes complement). $L$ is countably paracompact if for every sequence $L_n \in L$ and $L_n \cap L$ there exists $B_n' \in \delta(L)$ such that $B_n' \subseteq L$ and $B_n' \cap L = \emptyset$ for every $n$. A tau lattice is one that is closed under arbitrary intersections, and the tau lattice generated by $L$ is denoted by $\tau L$. A lattice $A(L)$ will denote the algebra generated by the lattice $L$.

Let $L_1, L_2$ be two lattices of sets $L_2 \subseteq L_1$, then $L_1$ semi-separates (ss) $L_2$ if for $L_1 \in L_1, L_2 \in L_2$ and $L_1 \cap L_2 = \emptyset$, then there exists an $A_1 \in A(L_1), A_2 \in A(L_2)$ such that $A_1 \cap A_2 = \emptyset$.

Let $I(L)$ denote the set of non-trivial two valued $\{0, 1\}$ finitely additive measures on the algebra generated by $L$, and let $I(\sigma*,L)$ denote those elements of $I(L)$ that are sigma-smooth on $L$, i.e. if $(L_n) \in I(L)$, $L_n \cap \emptyset$ and $\mu I(\sigma*,L), \lim \mu(L_n) = 0$, then $I(L)$ denotes those elements of $I(L)$ that are sigma-smooth on $L$. I.e. if $(A_n) \in A(L), A_n' \cap \emptyset, and \mu I(L) \lim \mu(A_n) = 0$ as $n \to \infty$. This is equivalent to countably additivity on $A(L)$. $IR(L)$ will stand for the measures on $A(L)$ that are $L$-regular on $L$, i.e. $\mu I(L) = \sup \mu(L) \in L$, $A(L) \cap L$ and $\alpha A(L)$. This is equivalent to being $L$-regular on $L$. $IR(\sigma,L)$ denotes the set of $\mu I(L)$ that are $\sigma$-smooth on $L$. The obvious relations hold, $I(L) \supseteq I(\sigma*,L) \supseteq I(L) \supseteq IR(L)$. The support of a measure $S(\mu), \mu I(L)$ is defined as $S(\mu) = \cap \{L \in I(L) | \mu(L) = 1\}$.

Let $L_1$ and $L_2$ be two lattices of sets of $X$ and $L_2 \subseteq L_1$, then $L_2$ is $L_1$ countably bounded (cb) if for $L_2 \in L_2$ and $L_2 \cap \emptyset$, then there exists $L_1 \in L_1, L_1 \cap \emptyset$ and $L_1 \cap L_2 = \emptyset$. $L$ is said to be disjunctive if for any $x \in X$ and $L \in L$, such that $x \in L$, there exists $L' \subseteq L$, such that $x \in L'$ and $L' \cap L = \emptyset$. $L$ is said to be regular if for $x \in X$ and $L \in L$, then there exists $L_1 \subseteq L_1 \subseteq L_2 \subseteq L$ and $L_1 \cap L_2 = \emptyset$. $L$ is said to be normal if for $L_1, L_2 \subseteq L$ and $L_1 \cap L_2 = \emptyset$, there exist $L_3, L_4 \subseteq L$ such that $L_3 \cap L_4 = \emptyset$. $L$ is said to be countably compact if for any $(L_n) \subseteq L$ and $\cap L_n = \emptyset$ $n = 1, 2, \ldots$, then there exists a finite subindexing $\cap L_n = \emptyset n = 1, 2, \ldots$ to $L$. $L$ is said to be $T_1$ if for $x, y \in X$ there exist $L_1 \subseteq L_2 \subseteq L$ such that $x \in L_1, y \in L_1'$ and $y \in L_2' \subseteq x \in L_2'$.

The lattice $V(L)$ (see below for definitions) in the space $I(L)$ is regular. The topology of closed sets $\tau V(L)$ in $I(L)$ is $T_1$. The lattice of sets $V(L)$ in $I(L)$ is disjunctive.
Note: For $g, g_1 \in \mathcal{I}(L)$, we write $g < g_1 (L)$ if $g(L) < g_1(L)$ for all $L \in L$.

We now note some measure equivalences of topological properties: 1) $L$ is disjunctive iff for all $x \in X$, $\mu_x \in \mathcal{I}(\sigma, L)$ and $\mu_x$ is the point mass measure, i.e. $\mu_x(A) = 1$ if $x \in A$, otherwise $\mu_x(A) = 0$. 2) $L$ is regular iff $\mu_1(L)$ and $\mu_2(L)$ implies $S(\mu_1) = S(\mu_2)$. 3) $L$ is normal iff $\mu_1 \leq \mu(L)$ and $\mu_1 \mu_2 \in \mathcal{I}(L)$ implies that $\mu_1 \leq \mu_2(L)$ and $\mu_2 \mu_1 \in \mathcal{I}(L)$.

We now note some measure equivalences of topological properties: 1) $L$ is disjunctive iff for all $x \in X$, $\mu_x \in \mathcal{I}(\sigma, L)$ and $\mu_x$ is the point mass measure, i.e. $\mu_x(A) = 1$ if $x \in A$, otherwise $\mu_x(A) = 0$. 2) $L$ is regular iff $\mu_1 \leq \mu_2(L)$ and $\mu_1 \mu_2 \in \mathcal{I}(L)$ implies $\mu_1 \leq \mu_2(L)$ and $\mu_2 \mu_1 \in \mathcal{I}(L)$.

We now prove a result that will be useful in the sequel.

Theorem 2.2: Let $L$ be normal and countably paracompact, then if $\mu \in \mathcal{I}(\sigma, L)$ there exists a unique $\mu \in \mathcal{I}(\sigma, L)$ st $\mu \leq \mu_1(L)$.

Proof: Let $\mu \in \mathcal{I}(\sigma, L)$ and $\mu \leq \mu_1(L)$. Then we must prove $\mu \in \mathcal{I}(\sigma, L)$. Let $A \in L$ and $A_n \downarrow \emptyset$. Since $L$ is countably paracompact there exists $(B_n) \downarrow \emptyset$ and $B_n \supseteq A_n$ for every $n$. Since $B_n \supseteq A_n$ and $L$ is normal and $A_n$ is normal, there exists $C_n, D_n \in L$ st $C_n \supseteq A_n$ and $D_n \supseteq B_n$ st $D_n \cap C_n = \emptyset$. Then $B_n \supseteq D_n \supseteq C_n \supseteq A_n$ and we can assume without loss of generality that these inclusions hold with $D_n \uparrow \emptyset$. Then $\mu_1(A_n) \leq \mu_1(C_n) \leq \mu(C_n) \leq \mu(D_n)$ and since $B_n \uparrow \emptyset$ and $\mu \in \mathcal{I}(\sigma, L)$ imply that $\mu(D_n) = 0$ as $n \to \infty$. Then $\mu_1(A_n) = 0$ as $n \to \infty$, and $\mu \in \mathcal{I}(\sigma, L)$. Uniqueness follows from normality.

Next we consider various sets of measures defined on the algebra generated by a lattice $L$. For example consider $\mathcal{I}(L), \mathcal{I}(\sigma, L), \mathcal{I}(L), \mathcal{I}(\sigma, L), \mathcal{I}(L), \mathcal{I}(\sigma, L)$. Denote such sets by $I$. Also consider the collection of sets $\mathcal{H}(L)$ where $\mathcal{H}(L) = \{ \mu \in \mathcal{I}(L) : \mu(L) = 1 \}$. Then the following hold: a) $\mathcal{H}(A \vee B) = \mathcal{H}(A) \cup \mathcal{H}(B)$ for $A, B \in \mathcal{L}$. b) $\mathcal{H}(A) \cap \mathcal{H}(B) = \mathcal{H}(A \cap B)$ for $A, B \in \mathcal{L}$. c) $\mathcal{H}(A) = \mathcal{H}(A)$ for $A \in \mathcal{L}$. d) If $A \supseteq B$ then $\mathcal{H}(A) \supseteq \mathcal{H}(B)$ for $A, B \in \mathcal{L}$. e) If $L$ is disjunctive (if necessary) and $\mathcal{H}(A) \supseteq \mathcal{H}(B)$ then $A \supseteq B$ for $A, B \in \mathcal{L}$.
We will assume in discussing $\mathcal{H}(L)$ for convenience, that $L$ is disjunctive, although it will be clear that this assumption is not always needed.

If $\mu_1(L)$ then define a measure on $\Lambda(\mathcal{H}(L))$ by $\mu_1(\mathcal{H}(A)) = \mu(A)$ for $A \in A(L)$. Conversely if $\mu_1(\mathcal{H}(L))$ define a measure on $\Lambda(L)$ by $\mu(A) = \mu(\mathcal{H}(A))\mathcal{H}(A) \in A(\mathcal{H}(L))$. Then the following hold:

Theorem 2.3: If $L$ is disjunctive (if necessary) then there is a 1-1 correspondence between the sets $I(L)$ and $I(\mathcal{H}(L))$ given by $\mu_1 \leftrightarrow \mu_1^\wedge$. Further $\mu_1(L)$ is $\sigma$-smooth or regular iff $\mu_1(\mathcal{H}(L))$ is $\sigma$-smooth or $\mathcal{H}(L)$ regular.

If $I = I(L)$ we let $H(L) = V(L)$.

If $I = I(\sigma, L)$ we let $H(L) = V(\sigma, L)$.

If $I = I(R, L)$ we let $H(L) = W(L)$.

If $I = I(\sigma, L)$ we let $H(L) = W(\sigma, L)$.

These sets are topologized by taking $H(L), H(L) \in J(L)$ as a basis for the closed sets, and will be referred to as generalized Wallman spaces.

3 THE SPACES $IR(L)$ AND $I(L)$

In this section we investigate a variety of conditions which are equivalent to $IR(L) = I(L)$ both abstractly and from a topological point of view with respect to the space $I(L), \tau V(L)$. This will be useful for our subsequent analysis of $I(\sigma, L)$ as well as being interesting in its own right.

Theorem 3.1: Let $L$ be a lattice of subsets of $X$, then the following are equivalent: a) $IR(L) = I(L)$

b) $IR(L') = IR(L)$

c) $V(L)$ in the space $I(L)$ is regular

d) The topology of closed sets $\tau V(L)$ in $I(L)$ is $T_1$

e) The lattice of sets $V(L)$ in $I(L)$ is disjunctive

f) $L$ is an algebra.

Proof: We show first that $IR(L) = I(L)$ iff $IR(L') = IR(L)$.

If $IR(L) = I(L)$ and if $\mu_1 IR(L')$, then $\mu_1 E(L)$ and thus $\mu_1 IR(L)$. Also if $\mu_1 IR(L)$ then $\mu_1 E(L')$ and there exists $\mu_1^\wedge E(L')$ st $\mu_1 \leq 1 \ (L')$. But $\mu_1 E(L) = IR(L)$ and therefore $\mu_1 = \mu_1 E(L')$. Conversely if $IR(L') = IR(L)$ let $\mu_1 E(L)$, then there exists $\mu_1 E(L) st \mu_1 \leq 1 \ (L')$ or $1 \leq \mu_1 (L')$. But $1 E(L')$, thus $\mu_1 = \mu_1^\wedge E(L')$ and $I(L') = IR(L')$.

Next we wish to show $IR(L) = I(L)$ iff the lattice $V(L)$ in the space $I(L)$ is regular.

Let $IR(L) = I(L)$ assume that $V(L)$ is not regular then there exists $V(L) \in V(L) \mu_1 E(L)$ st $\mu_1^\wedge E(L) = \{ V(L') \} V(L) = \{ V(L') \} V(L)$ or $\mu V(L)$ has the finite intersection property and there exists $\mu_1^\wedge E(L) st \mu_1^\wedge (V(L')) = 1$ and $\mu \leq \mu_1 (V(L'))$. Projecting down $\mu \leq 1 (L')$, and since $IR(L) = I(L)$ $\mu_1 = \mu_1^\wedge$. Then projecting upward $\mu_1^\wedge = \mu^\wedge$, $\mu^\wedge (V(L')) = 1$ and $\mu^\wedge (V(L)) = \mu_1^\wedge (V(L)) = 1$.

Conversely let $V(L)$ be regular in $I(L)$ let $\mu_1 E(L), \mu_1 L, \mu_1 (L) = 1$. Therefore $\mu_1 E(L)$. Since $V(L)$ is regular there exists $V(L) \in V(L) \mu_1 E(L)$ st $\mu_1^\wedge E(L) = \{ V(L') \} V(L)$ or $\mu E(V(L'))$. But this implies that $\mu_1 E(L)$ and $L \geq L_2 \mu_1 (L_2) = 1$ and $\mu_1 E(L)$, therefore $IR(L) \geq I(L)$ and $I(L) = IR(L)$.

Next we show $IR(L) = I(L)$ iff the topology of closed sets $\tau V(L)$ in $I(L)$ is $T_1$.

$\tau V(L)$ is $T_1$ if $V(L) = T_1$. Assume that $IR(L) = I(L)$ let $\mu_1, \mu_2 E(L) = IR(L)$ and $\mu_1 \neq \mu_2$. Then there exists $L_1, L_2 E L$ st $\mu_1 (L_1) = 1, \mu_2 (L_1) = 0, \mu_2 (L_2) = 0, and \mu_2 (L_2) = 1$, which implies that $\mu_1 E V(L_2), \mu_2 E V(L_2), \mu_1 E V(L_1'), \mu_2 E V(L_1')$, i.e. $V(L)$ is $T_1$. 
Conversely let \( V(L) \) be \( T_1 \) and let \( \mu \in L \). If \( \mu \in IR(L) \) then there exists \( v \in R(L) \) and \( L_1 \in L \) such that \( \mu \in V(L_1') \), \( v \in V(L_1') \) and \( \mu \leq V(L) \). Since \( V(L) \) is \( T_1 \), there exists \( L_2 \in L \) such that \( \mu \in V(L_2') \) and \( \nu \in V(L_2') \), i.e. \( \mu(L_2) = 1 \) and \( v(L_2) = 0 \), a contradiction.

Next we show \( IR(L) = I(L) \) iff the lattice of sets \( V(L) \) in \( I(L) \) is disjunctive.

Assume that \( IR(L) = I(L) \). Let \( \mu \in L \) and suppose \( L \in L \mu \in V(L) \). Since \( IR(L) = I(L) \) there exists \( L_1 \in L \) st \( L_1 \supseteq L_1', \mu(L_1) = 1 \) and \( \mu \in V(L) \) and \( V(L) \cap V(L') = \emptyset \), thus \( V(L) \) is disjunctive.

Conversely let \( V(L) \) be disjunctive. Let \( \mu \in L \), let \( L \in L \) and let \( \mu(L') = 1 \) (and hence \( \mu \in V(L') \)). Since \( V(L) \) is disjunctive there exists a \( V(L_1) \in V(L) \) st \( \mu \in V(L_1) \) and \( V(L_1) \varcap V(L') = \emptyset \). But this implies that \( L_1 \supseteq L_1', \mu(L_1) = 1 \), and \( \mu \in I(L) \). Therefore \( IR(L) = I(L) \).

Finally, now we claim \( I(L) = IR(L) \) iff \( L \) is an algebra, i.e. \( L = L' \).

Let \( L \) be an algebra and \( \mu \in L \) then since \( L = L' \mu \) is trivially regular and \( IR(L) = I(L) \). Conversely let \( I(L) = IR(L) \) and assume that \( L = L' \), i.e. that \( L \) is not an algebra. Thus there exists a \( L_1 \in L \) st \( L_1 \in L \) and look at \( H = \{ L_1 \mid L_1 \supseteq L \text{ or } L_1 \supseteq L \} \). Then \( H \) has the hip and thus there exists \( \mu \in L \) st \( \mu(L_1) = 1 \), \( L \in H \). For \( \mu(L_1) = 1 \), \( L \in L \) implies that \( L_1 \) does not contain \( L \) or \( L' \). Thus there exists \( \mu \in L \) st \( \mu(L_1) = 1 \) \( L \subseteq L \) (L) and also a \( \mu_2 \in L \) st \( \mu_2(L_1) = 1 \) and \( \mu \leq \mu_2(L) \). But since \( I(L) = IR(L) \) and this implies from above that \( IR(L') = IR(L) \mu_1(L_1') = \mu_1(L_1') = 1 \) or \( \mu_1(L_1 \cap L_1') = 1 = \mu_1(\emptyset) = 0 \), a contradiction. \( L = L' \) and \( L \) is an algebra.

Note: It is well known even for abstract distributive lattices that \( I(L) = IR(L) \) iff \( L = L' \). (See Bourbaki [2], Huerta [6]).

Because of the importance of the last result we present an alternative approach which is of importance because of its relevance to lattice separation properties.

**Theorem 3.2:** Suppose \( L_1, L_2 \) are lattices of subsets of \( X \) st \( L_2 \supseteq L_1 \). If \( L_2 \) is disjunctive and \( L_1 \) is normal and if \( \psi: IR(L_2) \to IR(L_1) \) where \( \psi \) is the restriction map, i.e. \( \psi(v) = \mu \) the restriction of \( v \) to \( A(L_1) \), then \( L_1 \) semi-separates \( L_2 \).

**Proof:** Suppose \( L_1 \in L_1 \) and \( L_2 \in L_2 \) and \( L_1 \cap L_2 = \emptyset \). Then \( W_2(L_1) \cap W_2(L_2) = \emptyset \) and \( \psi(W_2(L_2)) \cap W_1(L_1) = \emptyset \) for if \( \mu = \psi(v) \) where \( v \in W_2(L_2) \), then \( v(L_2) = 1 \). Therefore \( \mu(L_1) = \psi(v(L_1)) = 0 \) and thus \( \mu \in W_1(L_1) \).

\[ \psi(W_2(L_2)) = \cap W_1(L_1) \] is an arbitrary index set. Now \( L_1 \supseteq L_2 \). This holds since \( W_2(L_2) \) is closed and thus compact \( (W_2(X) \supseteq W_2(L_2) \) and \( W_2(X) \) is compact). Also \( \psi \) is continuous since \( \psi^{-1}(W_1(L_1)) = W_2(L_2) \) and \( W_2(X) \) is normal which is equivalent to \( W_1(L_1) \) being \( T_2 \) by a known result (see Bourbaki [2]). Therefore since \( W_2(L_2) \) is compact and \( \psi \) is continuous then \( \psi(W_2(L_2)) \) is compact and since \( W_1(L_1) \) is \( T_2 \), \( \psi(W_2(L_2)) \) is closed and thus \( \psi(W_2(L_2)) = \cap W_1(L_1) \) is an arbitrary index set. Also since \( L_2 \) is disjunctive then since \( \psi: IR(L_2) \to IR(L_1) \) is well defined \( L_1 \) is also disjunctive. But this implies that \( L_1 \supseteq L_2 \). Thus \( \psi(W_2(L_2)) = \cap W_1(L_1) \) is an algebra.

**Corollary 3.1:** If \( L \) is a lattice of subsets of \( X \) st \( I(L) = IR(L) \) then \( L \) is a lattice.

**Proof:** Set \( L_1 = L \) and \( L_2 = A(L) \), since \( I(L) = IR(L) \), \( L \) is normal. Then the hypotheses of the theorem hold, thus \( L \) is an algebra. Let \( L \cap L_1 = \emptyset \), then since \( L \subseteq A(L) \) and \( L \subseteq A(L) \), this implies that \( L \subseteq L \), i.e. \( L \) is an algebra.

Note: Suppose \( L_2 \) is disjunctive and \( cc \) then \( IR(\sigma, L_2) = IR(L_2) \) and if \( v \in IR(L_2) \) then \( \mu = \psi(v) \in (\sigma, L_1) \), and if \( L_1 \) is a delta lattice and \( \Delta(L_1) \supseteq (\sigma, L_1) \) then \( \mu \in IR(\sigma, L_1) \), in which case if \( L_1 \) is also normal then \( L_1 \) is a lattice by Theorem 3.2.
Another application arises if $L_2$ is $L_1$ cb and $L_1$ is cc then $IR(\sigma L_2)=IR(L_2).$ If $L_2$ is disjunctive and $L_1$ is a delta lattice, $\sigma(L_1)\supseteq S(L_1)$ and $L_1$ is normal, then theorem 3.2 can be applied and $L_1 \cong L_2.$

4. THE SPACES $I(\sigma*,L)$ AND $IR(\sigma,L)$

In this section we wish to consider, analogous matters concerning the spaces $I(\sigma*,L)$ and $IR(\sigma,L)$ to those considered earlier for $I(L)$ and $IR(L).$ First we obtain conditions when $I(\sigma*,L)=IR(\sigma,L)$ implies $L$ is an algebra. In this connection we introduce a definition.

Definition 4.1 The lattice of subsets of $X$ is almost countably compact (acc) if $\mu IR(\ell')$ implies $\mu I(\sigma*,\ell).$

Remark: Clearly $L$ cc implies $L$ acc. It is easy to show that if $L$ is normal and countably paracompact then $L$ acc implies $L$ cc: Namely let $\mu I(\ell)$ then $\mu I(L')$ and also there exists a $\mu I(\ell')$ st $\mu \subseteq \mu_1 L' \cup \mu_2 L.$ But $L$ acc implies that $\mu_1 I(\sigma*,L')$ and $L$ normal and cp implies there exists $\mu_2 I(\sigma,\ell)$ (see introduction) st $\mu_1 \subseteq \mu_2 (L)$ and thus $\mu \subseteq \mu_2 L.$ Thus $\mu I(\sigma*,L)$ and $L$ is cc.

Theorem 4.1: If $I(\sigma*)=IR(\sigma,L)$ and if $L$ is acc then $L$ is an algebra.

Proof: Let $\mu I(\ell)$ then there exists a $\mu I(\ell')$ st $\mu \subseteq \mu_1 L'.$ But $L$ acc, therefore $\mu_1 I(\sigma*,L)$.

Theorem 4.2: Consider the set $I(\sigma*,L),$ then the lattice $V(\sigma,L)$ in $I(\sigma*,L)$ is regular iff $IR(\sigma,L)$ is regular.

Proof: Assume that $I(\sigma*,L)=IR(\sigma,L)$ and that $V(\sigma,L)$ is not regular. Then there exists $\mu I(\sigma*,L)$ and $V(\sigma,L)E V(\sigma,L)$ such that $\mu V(\sigma,L)$ and $H=(V(\sigma,L)) \supseteq V(\sigma,L)$ or $\mu V(\sigma,L)).$ Then $\mu V(\sigma,L)$ st $\mu \subseteq \mu_1 V(\sigma,L)$ and $\mu_1 \subseteq \mu_2 V(\sigma,L).$ Then $\mu I(\sigma*,L)$ and projecting downward, $\mu \subseteq \mu_1 V(\sigma,L)$ and $\mu_1 \subseteq \mu_2 V(\sigma,L).$ Then $I(\sigma*,L)=IR(\sigma,L)$, therefore $\mu_1 \subseteq \mu_1 V(\sigma,L)$ and $\mu_1 \subseteq \mu_2 V(\sigma,L).$

Theorem 4.3: Suppose $xL\subseteq E\subseteq A(L)\subseteq L$ then if $xL$ is L cb or more generally if $E$ (and thus $A(L)$) is L cb and a) $S(L)\supseteq \sigma(L)$ (in particular if $\rho(L)=\sigma(L)$) and $L$ is delta or b) If $L$ is complement generated (and not necessarily delta) then $\rho(\sigma(x))=I(\sigma*,L).$

Proof: Let $\mu I(\sigma*,L).$ Since $A(L)$ is L countably bounded $I(\sigma,L)=I(\sigma*,L).$ Now let $L$ be cg and $\mu(L')=1,$ then $L=\cap L_i =\cap L_i \cap L=\cap L_i =\cap L_i.$ Then since $\mu I(\sigma,L),$ $0=\lim_{L_i=\cap L_i \cap L=\cap L_i} \mu(\cap L_i) =0$ where $i=1,2,..N$ and therefore $\mu(\cap L_i)=0$ where $i=1,2,..N$ for some $N.$ Since $L\supseteq \cap L_i$ $\mu I(\sigma,L)$ and $I(\sigma*,L)=IR(\sigma,L).$

Suppose instead that $L$ is delta and $S(L)\supseteq \sigma(L)$ in particular $\rho(\sigma(L))=\sigma(L).$ Since $A(L)$ is countably bounded by $L I(\sigma*,L)=I(\sigma,L).$ Consider $\sigma*$ the outer measure induced by $\sigma$ and its restriction to the $\sigma*$-measurable sets. Then the $\sigma*$-measurable sets include $\sigma(L)$ and thus $A(L),$ and $\sigma*$ is delta regular on such sets by the hypotheses $S(L)\supseteq \sigma(L)$ (or more particularly $\sigma(L)\supseteq \rho(L)).$ Since $L$ is delta this implies that $\mu I(\sigma,L)$ and $I(\sigma*,L)=IR(\sigma,L).$
Note: If \( I(\sigma^*, L) = IR(\sigma, L) \) then \( IR(\sigma, L^*) = IR(\sigma, L) \) and \( \mu_x IR(\sigma, L) \) and \( \mu_x IR(\sigma, L^*) \). Thus \( L \) and \( L^* \) is disjunctive, thus \( F \geq L' \) and \( \tau L \) contains \( L \) and \( L' \), which implies that \( L \) is contained in the algebra of closed-open sets determined by the topology of \( L \) on \( X \).

Definition 4.2: A lattice \( L \) is said to be prime complete if for any \( \mu \in I(\sigma^*, L) \) \( S(\mu) \neq \emptyset \).

Theorem 4.4: If \( L \) is disjunctive, and \( W(\sigma, L) \) in \( IR(\sigma, L) \) is prime complete, then for any \( \mu \in I(\sigma^*, L) \), there exists a \( \mu_1 \in IR(\sigma, L) \) s.t. \( \mu \leq \mu_1 \) (L).

Proof: Since \( L \) is disjunctive there exists a one to one correspondence between measures on \( X, L \) and \( IR(\sigma, L), W(\sigma, L) \). Thus let \( \mu \in I(\sigma^*, L) \) then \( \mu \in I(\sigma^*, W(\sigma, L)) \) and since \( W(\sigma, L) \) is prime complete \( S(\mu) \neq \emptyset \), then there exists a \( \{ \mu_1 \} \in S(\mu) \mu_1 \in IR(\sigma, L) \). Further if \( \mu (L) = 1 \) \( \mu \in L \) then \( \mu (W(\sigma, L)) = \mu_1 (L) = 1 \) and since \( \mu_1 \in W(\sigma, L) \mu_1 (L) = 1 \) therefore \( \mu \leq \mu_1 \) (L) with \( \mu_1 \in IR(\sigma, L) \).

Theorem 4.5: If a) \( L \) is disjunctive and \( W(\sigma, L) \) is prime complete or alternately b) \( L \) is normal and countably paracompact then if \( I(\sigma^*, L), V(\sigma, L) \) is T then \( I(\sigma^*, L) = IR(\sigma, L) \).

Proof: Let \( \mu \in I(\sigma^*, L) \) then by hypothesis a) or b) there exists a \( \nu \in IR(\sigma, L) \) s.t. \( \mu \leq \nu \). Then since \( I(\sigma^*, L), V(\sigma, L) \) is T1, there exists a \( L' \), \( \nu \in L \) s.t. \( \nu \in V(\sigma, L') \mu \in V(\sigma, L') \) which implies that \( \mu (L) = 1 \) \( \nu (L) = 0 \) a contradiction unless \( \mu = \nu \) and \( I(\sigma^*, L) = IR(\sigma, L) \).

Theorem 4.6: If \( I(\sigma^*, L), V(\sigma, L) \) is T1 then \( \mu_1 \in I(\sigma^*, L) \) implies that if \( \mu \neq \mu_1 \) then neither \( \mu \leq \mu_1 \) (L) or \( \mu_1 \leq \mu \) (L) holds. Conversely if neither \( \mu \leq \mu_1 \) (L) or \( \mu_1 \leq \mu \) (L) holds then \( I(\sigma^*, L), V(\sigma, L) \) is T1.

Proof: Let \( \mu, \mu \in I(\sigma^*, L) \) \( \mu \neq \mu_1 \). Since \( V(\sigma, L) \) is T1, this implies that there exists \( V(\sigma, L_1), V(\sigma, L_2) \in V(\sigma, L) \) s.t. \( \mu \in V(\sigma, L_1) \) \( \mu \in V(\sigma, L_2) \) \( \mu \in V(\sigma, L_2) \mu_1 \in V(\sigma, L_2) \mu_1 \in V(\sigma, L_2) \mu_1 \in V(\sigma, L_1) \mu_1 \in V(\sigma, L_1) \) and \( I(\sigma^*, L), V(\sigma, L) \) is T1.

Conversely suppose \( \mu, \mu_1 \in I(\sigma^*, L) \) \( \mu \neq \mu_1 \) and neither \( \mu \leq \mu_1 \) (L) or \( \mu_1 \leq \mu \) (L) holds. This implies that there exists \( L_1, L_2 \in L \) s.t. \( \mu (L_1) = 0 \) \( \mu (L_2) = 0 \) \( \mu (L_1) = 0 \) \( \mu (L_2) = 1 \) \( \mu (L_1) = 1 \) \( \mu (L_2) = 1 \). Thus neither \( \mu \leq \mu_1 \) (L) or \( \mu_1 \leq \mu \) (L) can hold.

Definition 4.3: Denote by \( \Pi(\sigma, L) \) the collection of premeasures that are sigma-smooth. A premeasure \( \mu \in \Pi(L) \) is defined on \( L \) and satisfies 1) \( p(\emptyset) = 0 \); 2) If \( \mu(A) = 1 \) \( B \in L \) then \( \mu(A \cap B) = 1 \); 3) If \( \mu(B) = 1 \) \( A \supseteq B \) then \( \mu(A) = 1 \). It is sigma-smooth if \( \{ A_n \} \downarrow \emptyset \) \( A_n \in L \) then \( \lim p(A_n) = 0 \) as \( n \to \infty \).

Definition 4.4: \( L \) is an \( \Pi(\sigma, L) \) iff for every \( \mu \in \Pi(\sigma, L) \) there exists a \( \mu \in IR(\sigma, L) \) s.t. \( \mu \leq \mu \) (L).

Theorem 4.7: Let \( L \) be an \( \Pi(\sigma, L) \) lattice which is also a delta lattice and suppose \( I(\sigma^*, L) = IR(\sigma, L) \), then \( L \) is complemented.

Proof: Assume \( L \) is not complemented, then for some \( \leq L \), \( \leq L \). Consider \( H = \{ L \} \sim L \preceq L \}, L \preceq L \}. \) Then since \( L \) is delta, \( H \) has the countable intersection property since \( L \neq \emptyset \). Thus there exists a \( \mu \in \Pi(\sigma, L) \) associated with \( H \). Since \( L \) is \( \Pi(\sigma, L) \) lattice there exists \( \mu \in IR(\sigma, L) \) s.t. \( \mu \leq \mu \) (L). Also \( I(\sigma^*, L) = IR(\sigma, L) \) implies that \( IR(\sigma, L) = IR(\sigma, L) \).

Now if \( L \preceq L \), then \( L \preceq L \), since \( \leq L \). Thus \( \mu (L) = 1 \) since \( \leq L \) associated with \( \mu \) is an ultrafilter. But since \( \mu \leq \mu \) (L) \( \mu (L) = 0 \), a contradiction and hence \( L \) is complemented.

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