ABSTRACT. Let $\beta G$ be the Stone-Cech compactification of a group $G$, $A^G$ the set of all almost periodic points in $\beta G$, $K^G = \text{cl}[\bigcup \{\text{supp}\mu_\varphi : \varphi \in \text{LIM}(G)\}]$ and $R^G$ the set of all recurrent points in $\beta G$. In this paper we will study the relationships between $K^G$ and $R^G$, and between $A^G$ and $R^G$. We will show that for any infinite elementary amenable group $G$, $A^G \subseteq R^G$ and $R^G - K^G \neq \emptyset$.

Key words and Phrases: Topological group, recurrent point, elementary amenable groups.

1980 AMS subject classification code. 43A60.

1. INTRODUCTION

Let $G$ be a discrete group, $M(G)$ the Banach space of all bounded real valued functions on $G$ with the supremum norm, and $M(G)^*$ the conjugate Banach space on $M(G)$. A group $G$ is called left amenable if there exists a mean on $M(G)$ which is left invariant [4]. Denote by LIM($G$) the set of left invariant means on $M(G)$. By using Riesz representation theorem, then there exists a bounded regular Borel measure $\mu_\varphi$ on $\beta S$ such that

$$\varphi(f) = \int_{\beta S} \overline{f}d\mu_\varphi, \quad f \in M(G).$$

For a subset $H$ of a left amenable group $G$ define the upper density as follows:

$$\overline{d}(H) = \sup \left\{ \varphi(\chi_H) : \varphi \in \text{LIM}(G) \right\} = \sup \left\{ \mu_\varphi(\overline{H}) : \varphi \in \text{LIM}(G) \right\}$$

where $\chi_H$ is the characteristic function of $H$.

Put $K^G = \text{cl}[\bigcup \{\text{supp}\mu_\varphi : \varphi \in \text{LIM}(G)\}]$. 
DEFINITIONS:

(1) For each $A \subseteq G$, let $\hat{A} = e^G A - G$. The set $\hat{A}$ is closed and open in $\hat{G} = \beta G - G$. Then a point $w \in \hat{G}$ is said to be almost periodic if for every neighbourhood $U$ of $w$ there exist $A, K \subseteq G$ with $K$ finite, $Aw \subseteq U$, and $G = KA$. Denote by $A^G$ the set of all almost periodic points in $\beta G$.

(2) $w \in \beta G$ is recurrent if whenever $U$ is a neighbourhood of $w$ then the set $\{x \in G : xw \in U\}$ is infinite. Let $R^G$ be the set of all recurrent points in $\beta G$.

(3) $w \in \beta G$ is discrete if its orbit $0(w)$ is discrete with respect to the subspace topology of $\beta G$. Let $D^G$ be the set of all such points in $\beta G$.

(4) $w \in \beta G$ is strongly discrete if there exists a neighbourhood $U$ of $w$ such that $x \cup y \cap U = \emptyset$ where $x, y \in G, x \neq y$. The set of all strongly discrete points will be denoted by $D_{SD}^G$. Let $WR^G = \beta G - SD^G$, the set of all weak recurrent points of $\beta G$.

If $G$ is a finite group, then $A^G = D^G = K^G = G = \beta G$ and $R^G = WR^G = \emptyset$. Therefore, we are only interested in infinite groups. For the remainder of this paper, $G$ stands for an infinite group.

LEMMA 1.1. (1) $A^G \subseteq R^G \subseteq WR^G$, $SD^G \subseteq D^G$ and $D^G = \beta G - R^G$.

(2) If $G$ is amenable, then $A^G \subseteq K^G \subseteq WR^G$. [4]

As mentioned in Day [3], the family of amenable groups is closed under the following four standard processes of constructing new groups from given ones: (a) subgroup, (b) factor group, (c) group extension, (d) expanding union (or direct limit). Denote by $E_n$ the family of elementary amenable groups. It is the smallest family of groups containing all finite groups and all abelian groups, and is closed under (a) - (d).

As pointed out by Chou [2], the groups in $E_n$ can be constructed from abelian groups and finite groups by applying processes (c) and (d) only. Moreover, every periodic group in $E_n$ is locally finite, and since it is known that every infinite locally finite group contains an infinite abelian subgroup (Robinson [5; p. 95]), we have the following:

LEMMA 1.2 (Chou). If $G$ is an infinite group in $E_n$, then $G$ contains an infinite abelian subgroup.
2. $R_0$-SETS AND RECURRENT POINTS:

In this section we will show that $\mathcal{A}^G \subseteq \mathcal{R}^G$ and $\mathcal{R}^G - K^G \neq \emptyset$, for any infinite elementary amenable group $G$.

DEFINITION. A subset $A$ of an amenable group $G$ is called $R_0$-set if:

$($R_0 \cdot 1$) \quad d(A) = 0$, \quad and

$($R_0 \cdot 2$) \quad \overline{A} \cap \mathcal{R}^G \neq \emptyset$.

PROPOSITION 2.1: If an amenable group $G$ contains an $R_0$-set, then $\mathcal{A}^G \subseteq \mathcal{R}^G$ and $\mathcal{R}^G - K^G \neq \emptyset$.

PROOF: If $A$ is an $R_0$-set in $G$, then by $(R_0 \cdot 2)$ there exists $w \in \overline{A} \cap \mathcal{R}^G$ since $d(A) = 0$, $w \notin K^G$. The proof is completed.

$R_0$-sets were first studied by Chou for the case $G = \mathbb{Z}$ where he showed that $\mathbb{Z}$ contains such sets (see Chou [1], P. 60, example 2). In this section we will show that every finite elementary amenable group contains $R_0$-sets.

THEOREM 2.2: Let $\{A_k\}$ be a sequence of subsets of a group $G$, and $\{g_k\}$ be a sequence of different elements of $G$ such that

1. $A_1 \supset A_2 \supset A_3 \supset \ldots$, and

2. $g_n A_{n+1} \subseteq A_n$ for all $n$. Then $\bigcap_{n=1}^{\infty} A_n$ contains a recurrent point.

PROOF. Let $\pi$ be the family of sequences of closed subsets of $\hat{G}$. Let $\pi$ be defined as follows: a sequence of closed subsets $\{F_n\}$ of $\hat{G}$ belongs to $\pi$ if for each $n \in \mathbb{N}$,

(i) $F_n \subseteq \hat{A}_n$,\n
(ii) $F_{n+1} \subseteq F_n$,\n
(iii) $g_n F_{n+1} \subseteq F_n$, and\n
(iv) $F_n \neq \emptyset$.

Note that $\pi$ is non-empty since $\{\hat{A}_n\} \in \pi$. $\pi$ can be ordered in the following natural way: $\{F_n\} \leq \{F'_n\}$ if and only if $F_n \subseteq F'_n$ for each $n$. It is easy to check that each chain in $\pi$ has a lower bound. Then using Zorn's lemma, $\pi$ has a minimal element $\{K_n\}$. Let $w \in \bigcap_{n=1}^{\infty} K_n$. We claim that $w$ is a recurrent point, by showing that if $\hat{A}$ is an open neighbourhood of $w$, then there exists infinitely many $g \in G$ such that $gw \in A$. 
Let $V = \bigcup_{g \in G} g \hat{A}$. Consider the sequence $\{K_n - V\}$. It satisfies (i), (ii), and using the fact that for $g \in G$, $gV = V$, one sees that $\{K_n - V\}$ satisfies (iii). Since $K_n - V \subset K_n$ and $\{K_n\}$ is minimal in $\pi$, $\{K_n - V\} \not\subset \pi$, therefore, $\{K_n - V\}$ does not satisfy (iv), i.e., there exists $n_0$ such that $K_{n_0} - V = \phi$ which implies $K_{n_0} \subset V = \bigcup_{g \in G} g \hat{A}$. Since $K_{n_0}$ is compact, there exists $a_1, a_2, \ldots, a_m \in G$ such that

$$K_{n_0} \subset \bigcup_{i=1}^m a_i \hat{A}. \quad (*)$$

For each $n \geq n_0, g_n \in G, K_{n+1} \subset K_n \subset K_{n_0}$, thus by $(*) g_n a_i \hat{A}$ for some $1 \leq i \leq m$. Then clearly, there exists $i_0, i_0 \leq m$ such that $g_n a_i \hat{A}$ for infinitely many $n$. Thus the set $\{g \in G : g \hat{A}\}$ is infinite. Therefore we $R^G$.

REMARK: When $G = Z$, the above theorem is contained in Chou [1]. The idea of the proof of the above theorem follows from [1, Proposition 3.1].

LEMMA 2.3 (Chou [1]): The additive group of integers $Z$ contains an $R_0$-set.

For prime number $p$, let

$$Z(p^n) = \{m/p^n : 0 \leq m < p^n, m \in Z, n = 0, n = 1, 2, \ldots\}$$

be the subgroup of $Q/Z$ generated by

$$\left\{ \frac{1}{p^n} : n = 1, 2, \ldots \right\}.$$

LEMMA 2.4: For each prime number $p$, $A(p^n)$ contains an $R_0$-set.

PROOF: Let $G = A(p^n)$. Then $G$ is a subgroup of $Q/Z$. Note that $G$ can be written as $G = \bigcup_{n=1}^\infty H_n$ where $H_1 \subset H_2 \subset \ldots$ and each

$$H_n = \left\{ 0, \frac{1}{p^n}, \frac{2}{p^n}, \ldots, p^n - \frac{1}{p^n} \right\}.$$  

is a cyclic group of order $p^n$. For convenience, we will write $H_n = (\frac{1}{p^n})$ (with the usual addition in $Q$ (mod $Z$)). Then following Chou's construction for $Z$ [1], we define a sequence of subsets $E_n$ in $G$ by induction as follows:

$$E_1 = \left\{ \frac{1}{p} \right\}, E_{n+1} = E_n \cup (E_n + \frac{1}{p^{(n+1)^2}})$$

Then

$$|E_n| = 2^{n-1}.$$
Let
\[ \bigcup_{n=1}^{\infty} E_n = \left\{ \frac{1}{p}, \frac{1}{p}, \frac{1}{p^4}, \frac{1}{p^9}, \frac{1}{p^9}, \frac{1}{p^9}, \frac{1}{p}, \frac{1}{p^4}, \frac{1}{p^9}, \frac{1}{p^9}, \frac{1}{p^9}, \frac{1}{p}, \frac{1}{p}, \frac{1}{p}, \frac{1}{p^4}, \frac{1}{p} \right\}. \]

Consider the \( F \)-sequence \( \{F_n\} \) in \( G \) where \( F_n = H_{n^2} \).

Then
\[ |F_n| = p^{n^2} \]
and
\[ \overline{d}(E) = \ell \lim_{n \to \infty} \frac{|E_n|}{|F_n|} = \ell \lim_{n \to \infty} \frac{2}{p^{n^2}} = 0. \]

Thus \( E \) satisfies \( (R_{0} \cdot 1) \).

We may choose a sequence of infinite subsets \( C_n \) in \( G \) such that
\[ C_1 = \left\{ \frac{1}{p}, \frac{1}{p}, \frac{1}{p^4}, \frac{1}{p^9}, \frac{1}{p^9}, \frac{1}{p^9}, \frac{1}{p}, \frac{1}{p^4}, \frac{1}{p^9}, \frac{1}{p^9}, \frac{1}{p^9}, \frac{1}{p}, \frac{1}{p}, \frac{1}{p}, \frac{1}{p^4}, \frac{1}{p} \right\}, \]
\[ C_2 = \left\{ \frac{1}{p}, \frac{1}{p^4}, \frac{1}{p^9}, \frac{1}{p^9}, \frac{1}{p^9}, \frac{1}{p^9}, \frac{1}{p}, \frac{1}{p^4}, \frac{1}{p^9}, \frac{1}{p^9}, \frac{1}{p^9}, \frac{1}{p}, \frac{1}{p}, \frac{1}{p}, \frac{1}{p^4}, \frac{1}{p} \right\}. \]
\[ C_3 = \left\{ \frac{1}{p}, \frac{1}{p^4}, \frac{1}{p^9}, \frac{1}{p^9}, \frac{1}{p^9}, \frac{1}{p^9}, \frac{1}{p}, \frac{1}{p^4}, \frac{1}{p^9}, \frac{1}{p^9}, \frac{1}{p^9}, \frac{1}{p}, \frac{1}{p}, \frac{1}{p}, \frac{1}{p^4}, \frac{1}{p} \right\}, \] and so on.

So
\[ E \supset C_1 \supset C_2 \supset \ldots \]
and
\[ C_{n+1} + p^{-n(n+1)} \subset C_n \text{ for all } n. \]

Then by theorem 2.2, \( E \) contains a recurrent point, and hence \( E \) satisfies \( (R_{0} \cdot 2) \).

**REMARK:** If \( X \) is a discrete set and \( Y \) is a subset of \( X \), then we will consider \( \beta Y \) as a subset of \( \beta X \). Indeed \( \beta Y \) is the closure of \( Y \) in \( \beta X \).

**LEMMA 2.5:** If an amenable group \( G \) contains an infinite subgroup \( H \) of infinite index, then \( H \) is an \( R_{0} \)-set in \( G \).
PROOF: Since \([G : H] = \infty\), the number of disjoint cosets \(\{tH\}\) is infinite. For any \(\varphi \in \text{LIM}(G)\), since \(\varphi(x_{tH}) = \varphi(x_{tH})\) if \(t_1H, t_2H, \ldots, t_nH\) are distinct cosets, then \(\varphi(x_{1H}, x_{2H}, \ldots, x_{nH}) = \varphi(x_{1H}) + \ldots + \varphi(x_{nH}) = n\varphi(x_{nH}) < 1\). Since \(n\) is arbitrary, \(\varphi(x_{nH}) = 0\) so \(d(H) = 0\). Thus \(H\) satisfies \((R_0 \cdot 1)\). To see that \(H\) satisfies \((R_0 \cdot 2)\) in \(G\), note that if \(w \in R^H\) and if \(U\) is a neighbourhood of \(w\) in \(\beta G\), then \(\cap U\) is also a neighbourhood of \(w\) and \(X' = \{h \in H : h \in U \cap H\}\) is infinite. Therefore \(\{g \in G : g \in U\} \cap X\) is infinite. Thus \(w \in R^G\). Since \(\phi \neq A^H \subset R^H, R^G \cap H \neq \phi\). Thus \(H\) satisfies \((R_0 \cdot 2)\) in \(G\).

**COROLLARY 2.6:** If an amenable group \(G\) is a direct product of infinitely many non-trivial subgroups \(\{G_{\alpha} : \alpha \in I\}\), then \(G\) contains an \(R_0\)-set.

**PROOF:** To prove that \(G\) contains an \(R_0\)-set, we construct an infinite subgroup \(H\) of \(G\) of infinite index with the above lemma. Indeed, write \(I = I_1 \cup I_2\) such that both \(I_1\) and \(I_2\) are infinite, then \(H = \pi \{G_{\alpha} : \alpha \in I_1\}\) is what we want.

**LEMMA 2.7:** If an infinite subgroup \(H\) of an amenable group \(G\) contains an \(R_0\)-set \(A\), then \(A\) is also an \(R_0\)-set in \(G\).

**PROOF:** This is quite obvious since whenever \(d_H(A) = 0\) then \(\bar{d}_G(A) = 0\) and \(R^H \subset R^G\).

**LEMMA 2.8:** If \(H\) is an amenable subgroup of an amenable group \(G\) such that \(G/H\) has an \(R_0\)-set \(A'\), then \(\theta^{-1}(A')\) is an \(R_0\)-set in \(G\) (\(\theta\) is the natural homomorphism of \(G\) onto \(G/H\)).

**PROOF:** Let \(A = \theta^{-1}(A')\) and \(G' = G/H\). Then for each \(\varphi \in \text{LIM}(G)\), \(\varphi(x_{\theta}) = \varphi(x_{\theta}) = 0\) since \(\theta^*\varphi \in \text{LIM}(G')\) and \(d_{G'}(A') = 0\). Then \(A\) satisfies \((R_0 \cdot 1)\).

The natural homomorphism \(\theta\) of \(G\) onto \(G'\) can be extended to a continuous mapping of \(\beta G\) onto \(\beta G'\). We will denote the extended mapping again by \(\theta\). It is not hard to check that \(\theta^{-1}(R^G) \subset R^G\), since \(A^- \cap R^G \neq \phi\), and \(A^- \cap R^G \neq \phi\) either. Thus \(A\) satisfies \((R_0 \cdot 2)\), and \(A\) is an \(R_0\)-set as wanted.

Before providing our main result, we will need some definitions and some structure theorems for abelian groups. For the proofs see [6].

**DEFINITION:** Let \(x \in G\) and \(n\) be an integer; \(x\) is divisible by \(n\) in case there is an element \(y \in G\) with \(ny = x\). A group \(G\) is divisible in case each \(x \in G\) is divisible by every \(n > 0\).
A subgroup $S$ of $G$ is **pure** in $G$ in case $nG \cap S = nS$ for every integer $n$.

If $G$ is a periodic group, then a subgroup $B$ of $G$ is a **basic** subgroup of $G$ in the following cases: (i) $B$ is the direct sum of cyclic groups; (ii) $B$ is pure in $G$; (iii) $G/B$ is divisible.

**THEOREM 2.9:**

1. Every periodic group $G$ contains a basic subgroup $B$.
2. Every periodic group $G$ is a direct sum of $p$-groups.
3. Every periodic group is an extension of a direct sum of cyclic groups by a divisible group.
4. Every divisible subgroup $D$ is a direct sum of copies of $Q$ and of copies of $Z(p^\infty)$.

We are now ready to state and prove the main results of this paper.

**THEOREM 2.10:** If $G$ is an infinite abelian group, then $G$ contains an $R_0$-set.

**PROOF:** We will consider two cases:

1. $G$: non-periodic. Then $G$ contains an infinite cyclic subgroup which can be regarded as the additive group of integers $Z$. Thus by lemmas 2.3 and 2.7, there exists an $R_0$-set.

2. $G$: periodic. Then by 2.9, $G$ contains a basic subgroup $B$, so that $B$ is a direct sum of cyclic groups. Here we have two subcases:

   (a) If $B$ is an infinite subgroup, by corollary 2.6 and lemma 2.7, there exists an $R_0$-set.

   (b) If $B$ is a finite subgroup, the question group $G/B$ is an infinite divisible that can be written as a direct sum of $\leq$ copies of $Z(p^\infty)$ (see 2.9). Consider one of these copies and apply lemmas 2.4 and 2.7 to get an $R_0$-set in $G$. Then the proof of the theorem is completed.

**THEOREM 2.11:** If $G$ is an infinite elementary amenable group, then $G$ contains an $R_0$-set.

**PROOF:** If $G$ is an abelian group, the theorem follows from the above one. If $G$ is not abelian, then by lemma 1.2, $G$ contains an infinite abelian subgroup $H$, and hence there exists an $R_0$-set $A$ in $H$. By lemma 2.7, $A$ is also an $R_0$-set in $G$.
THEOREM 2.12: If $G$ is an infinite elementary amenable group, then

$$R^G \supset A^G$$ and $$R^G - K^G \neq \emptyset.$$ 

The proof follows from the above theorem and proposition 2.1.

COROLLARY 2.13: If $G$ is an infinite elementary amenable group, then

$$R^G \subseteq W R^G.$$ 

This follows immediately from the above theorem and lemma 1.1.

CONJECTURE: Every infinite amenable group $G$ contains an $R_0$-set, and therefore

$$A^G \subseteq R^G$$ and $$R^G - K^G \neq \emptyset.$$ 

ACKNOWLEDGEMENT: The author would like to thank the referee for his comments.

REFERENCES


