ON A NONLINEAR DEGENERATE EVOLUTION EQUATION WITH STRONG DAMPING

JORGE FERREIRA
IM/UFRJ and Univ. Estadual de Maringá, Paraná
and
DUCIVAL CARVALHO PEREIRA
UFPA, Belém, Pará

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ABSTRACT. In this paper we consider the nonlinear degenerate evolution equation with strong damping,

(*) \[ K(x,t)u_t - \Delta u - \Delta u_t + F(u) = 0 \quad \text{in} \quad Q = \Omega \times ]0,T[ \]

where \( K \) is a function with \( K(x,t) \geq 0, K(x,0) = 0 \) and \( F \) is a continuous real function satisfying

(**) \[ sF(s) \geq 0, \quad \text{for all} \quad s \in \mathbb{R}, \]

\( \Omega \) is a bounded domain of \( \mathbb{R}^n \), with smooth boundary \( \Gamma \). We prove the existence of a global weak solution for (*).

KEY WORDS AND PHRASES. Weak solutions, evolution equation with damping.

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1. INTRODUCTION.

In this work we study the existence of global weak solutions for the degenerate problem

(1.1) \[
\begin{align*}
K(x,t)u_t - \Delta u - \Delta u_t + F(u) &= 0 \\
u(0) &= u_0 \\
(Ku')(0) &= 0 \\
u &= 0 
\end{align*}
\]

in the cylinder \( Q = \Omega \times ]0,T[ \) where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with smooth boundary, \( T > 0 \) is an arbitrary real number, \( \Sigma \) is a lateral boundary of \( Q \), \( F \) is a continuous real function such that \( sF(s) \geq 0 \), for all \( s \in \mathbb{R}, K : Q \to \mathbb{R} \) is a function such that \( K(x,t) \geq 0, (x,t) \in Q, K(x,0) = 0 \), \( \Delta \) is the Laplace operator and \( u' = \frac{du}{dt} \).

Equation (1.1) is a nonlinear perturbation of the wave equation. For \( n = 1 \) or \( n = 2 \), (1.1) governs the motion of a linear Kelvin solid (a bar if \( n = 1 \) and a plate if \( n = 2 \)) subject to no nonlinear elastic constraints, where \( K(x,t) \) is a mass density.
Problem (1.1) with $K(x,t) = 1$ without the term $-\Delta u'$ was studied by Strauss [1]. He proves the existence of global weak solutions and the asymptotic behavior as $t$ approaches to infinity. The global weak solutions for the equation

$$K_t(x,t)u + K_2(x,t)u' - Au + F(u) = 0 \tag{1.2}$$

with $K_1(x,t) \geq 0, K_2(x,0) \geq \alpha > 0$ and $K_3(x,t) \geq \beta > 0$ was studied by Maciel [2].

Problem (1.2) was also studied by Mello [3] for $F \in C^1(R), F(0) = 0, \int_0^1 F(\xi)d\xi \geq 0$, $F'$ dominated by $|s|^p, p > 0, K_2$ independent of $t$ non-zero initial data.

In [4] and [5], Larkin studied problem (1.2) with $F(u) = u^p$ and $F(u) = u^p u', p > 0$, respectively. In both cases the initial data are zero.

Problem (1.1) with $K(x,t) = 1$ was studied by Ang and Dinh [6] with $F \in C^1(R), F(0) = 0$ and $F' \equiv -C$ with $C > 0$ "small." They proved the existence of global weak solutions and the asymptotic behavior when $t$ approaches to infinity.

We denote by $(,\cdot), |\cdot|, (\cdot,\cdot), \|\cdot\|$ the inner and norm of $L^2(\Omega)$ and $H_0^1(\Omega)$, respectively, and $a(u,v) = \sum_{i=1}^{n} \int_{a}^{b} \frac{u_i}{n} \frac{v_i}{n} dx$ represents Dirichlet's form in $H_0^1(\Omega)$.

2. ASSUMPTIONS AND MAIN RESULTS.

We consider the following hypothesis:

(H.1) $F : R \to R$ is continuous with $sF(s) \geq 0, \forall s \in R$;

(H.2) $K \in C^1([0,T] : L^\infty(\Omega))$ with $K(x,t) \geq 0, (x,t) \in Q$ and $K(x,0) = 0$

(H.3) $|\frac{dF}{ds}| \leq \delta + C(\delta)K, \forall \delta > 0$ where $C(\delta)$ is a positive constant.

Then we have the following result:

**THEOREM 1.** Under hypothesis (H.1)-(H.3) if $G(s) = \int_0^s F(\xi)d\xi$ and $u_0 \in H_0^1(\Omega), G(u_0) \in L^1(\Omega)$ then there exists a function $u : [0,T] \to L^2(\Omega)$ such that:

$$u \in L^\infty(0,T : H_0^1(\Omega)) \tag{2.1}$$

$$u' \in L^\infty(0,T : H_0^1(\Omega)) \tag{2.2}$$

$$\sqrt{K(x,t)} u' \in L^2(0,T : L^2(\Omega)) \tag{2.3}$$

$$K(x,t)u' \in L^2(0,T : H_0^1(\Omega)) \tag{2.4}$$

$$\frac{d}{dt}(Ku',v) - (Ku',v) + a(u,v) + a(u',v) + (F(u),v) = 0 \quad \text{in} \quad D(0,T), \forall v \in H_0^1(\Omega) \tag{2.5}$$

$$u(0) = u_0 \tag{2.6}$$

$$(Ku')(0) = 0 \tag{2.7}$$

We divide the proof in two parts:

i) We consider $F$ Lipschitzian and derivable except on a finite number of points with $sF(s) \geq 0, \forall s \in R$. 

ii) We consider $F$ continuous with $F(s) \geq 0$, $\forall s \in \mathbb{R}$ and approximate $F$ by a sequence $(F_n)_{n \in \mathbb{N}}$ such that $sF_n(s) \geq 0$, $\forall s \in \mathbb{R}$, $\forall n \in \mathbb{N}$, with $F_n \to F$ uniformly on bounded sets of $\mathbb{R}$.

### 2.1 LIPSCHITZIAN CASE

We have the following result:

**THEOREM 2.** Let $F : \mathbb{R} \to \mathbb{R}$ be such that $sF(s) \geq 0$, Lipschitzian and derivable except on a finite number of points. Let $u_0 \in H^1_0(\Omega) \cap H^2(\Omega)$ with $G(u_0) \in L^1(\Omega)$, where $G(s) = \int_0^s F(\xi)\,d\xi$.

Then there exists a unique function $u : Q \to \mathbb{R}$ satisfying:

\[
\begin{align*}
&u \in L^\infty(0, T; H^1_0(\Omega)) \\
&u' \in L^2(0, T; H^1_0(\Omega)) \\
&K(x, t)u'' - Au - \Delta u' + F(u) = 0 \text{ in } L^2(0, T; H^{-1}(\Omega)) \\
&u(0) = u_0, \quad u'(0) = 0.
\end{align*}
\]

**PROOF.** Let $(w_v)_{v \in \mathbb{N}}$ be a basis of $H^1_0(\Omega) \cap H^2(\Omega)$ and $V_m = [w_1, ..., w_m]$ the subspace generated by the $m$ first vectors of $(w_v)_{v \in \mathbb{N}}$.

### 2.1.1 APPROXIMATION PERTURBED PROBLEM

Fix $\varepsilon > 0$ and for each $m \in \mathbb{N}$ consider a function of the form

\[u_m(t) = \sum_{j=1}^{m} g_j(t)w_j\]

such that $u_m(t)$ is a solution of the problem:

\[
\begin{align}
(K + \varepsilon)u_m'' + a(u_m, w) + a(u_m', w) + (F(u_m), w) &= 0, \quad \forall w \in V_m \tag{2.13} \\
u_m(0) = u_m \to u_0 \quad \text{strongly in } H^1_0(\Omega) \cap H^2(\Omega) \tag{2.14} \\
u_m'(0) &= 0 \tag{2.15}
\end{align}
\]

By Caratheodory's theorem, $u_m(t)$ exists on $[0, T_m]$, $T_m < T$. The a priori estimates will allow us to extend $u_m(t)$ to whole interval $[0, T]$.

### 2.1.2 A PRIORI ESTIMATES

1) Consider $w = u_m'(t)$ in (2.13). We obtain

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \left[ K(u_m^2) + \varepsilon |u_m'|^2 + \|u_m^2\|^2 + 2 \int_{\Omega} G(u_m)dx \right] + |u_m'|^2 - \frac{1}{2} \left[ \frac{\partial K}{\partial t}, u_m^2 \right]
\end{align*}
\]

Integrating from 0 to $t \leq T_m$ and using (H.3) we get:

\[
\begin{align*}
(K(u_m^2) + \varepsilon |u_m'|^2 + \|u_m^2\|^2 + 2 \int_{\Omega} G(u_m)dx + 2 \int_0^t |u_m'|^2 ds
\end{align*}
\]

\[
\leq \|u_m^2\|^2 + 2 \int_{\Omega} G(u_m)dx + \int_0^t \left[ |u_m'|^2 + C(\delta)(K(u_m^2) + |u_m'|^2) \right] ds
\]
By (2.14) and because \( G(u_0) \in L'(\Omega) \) we have:

\[
\int_{\Omega} G(u_{\text{m}}) \, dx = \int_{\Omega} G(u_0) \, dx
\]  
(2.16)

By (2.14)-(2.16) and Gronwall's inequality, it follows that:

\[
(K, u_{\text{m}}^2) + \varepsilon |u_{\text{m}}'|^2 + \|u_{\text{m}}\|^2 + 2 \int_{\Omega} G(u_{\text{m}}) \, dx + (2 - \delta) \int_0^t \|u_{\text{m}}\|^2 \, ds \leq M
\]

where \( M \) is a positive constant independent of \( \varepsilon, m, t, C \) is a positive constant such that \(|\nu|^2 \leq C \|\nu\|^2\) and \( \delta < \min \left(2, \frac{2}{\varepsilon} \right) \). Thus

\[
\left( K u_{\text{m}}' \right) \text{ is bounded in } L^\infty(0, T; L^2(\Omega))
\]  
(2.17)

\[
(u_{\text{m}}) \text{ is bounded in } L^\infty(0, T; H^1_0(\Omega))
\]  
(2.18)

\[
(u_{\text{m}}^2) \text{ is bounded in } L^2(0, T; H^1_0(\Omega))
\]  
(2.19)

\[
(\sqrt{\varepsilon} u_{\text{m}}') \text{ is bounded in } L^2(0, T; L^2(\Omega))
\]  
(2.20)

II) Since \( F \) is Lipschitzian and derivable except on a finite number of points of \( \mathbb{R} \), we can differentiate with respect to \( t \) to obtain

\[
\left[ \frac{\partial K}{\partial t} u_{\text{m}}' \right] + (K u_{\text{m}}') + \varepsilon (u_{\text{m}}')^2 + \|u_{\text{m}}\|^2 + 2 \int_{\Omega} G(u_{\text{m}}) \, dx + (2 - \delta) \int_0^t \|u_{\text{m}}\|^2 \, ds + 2F'(u_{\text{m}}) u_{\text{m}}'' - 0
\]  
(2.21)

Taking \( w = u_{\text{m}}'(t) \) in (2.21), we get

\[
\frac{d}{dt} [(K, u_{\text{m}}') + \varepsilon |u_{\text{m}}'|^2 + \|u_{\text{m}}^2\|] + 2 \int_{\Omega} G(u_{\text{m}}) \, dx + (2 - \delta) \int_0^t \|u_{\text{m}}\|^2 \, ds + 2F'(u_{\text{m}}) u_{\text{m}}'' = 0
\]  
(2.22)

But

\[
2F'(u_{\text{m}}) u_{\text{m}}'' \leq 2|F'(u_{\text{m}})| |u_{\text{m}}''| \leq 2\beta |u_{\text{m}}''| |u_{\text{m}}''|\n\]  
(2.23)

where \( \beta \) is a positive constant.

Integrating (2.22) from 0 to \( t \) and using (2.14)-(2.15), (2.23) and (H.3), it follows that

\[
(K, u_{\text{m}}^2) + \varepsilon |u_{\text{m}}'|^2 + \|u_{\text{m}}^2\| + (2 - \delta) \int_0^t \|u_{\text{m}}\|^2 \, ds \leq \varepsilon |u_{\text{m}}(0)|^2 + C_1 \int_0^t \|u_{\text{m}}\|^2 + (K, u_{\text{m}}^2) \, ds
\]  
(2.24)

where \( C_1 \) is a positive constant.

Now, we are going to estimate the term \( \varepsilon |u_{\text{m}}(0)|^2 \). Consider \( t = 0 \) in (2.13), and \( w = u_{\text{m}}(0) \). Then we get

\[
\varepsilon |u_{\text{m}}(0)|^2 \leq |\Delta u_{\text{m}}| + |F(u_{\text{m}})| \leq C
\]  
(2.25)

where \( C \) is a positive constant independent of \( \varepsilon, m \) and \( t \).
By (2.24), (2.25) and Gronwall's inequality, there exists a positive constant $M_1$, independent of $\varepsilon$, $m$, and $t$, such that:

$$
(K, u^{\varepsilon}_{\infty}) + \varepsilon |u_{\infty}'|^2 + \|u_{\infty}'\|^2 + (2 - \delta) \int_0^t \|u_{\infty}'\|^2 ds \leq M_1
$$

So,

$$
\left(\frac{1}{k} u_{\infty}'\right) \text{ is bounded in } L^\infty(0, T; L^2(\Omega)) 
$$

$$
(\sqrt{\varepsilon} u_{\infty}') \text{ is bounded in } L^\infty(0, T; L^2(\Omega)) 
$$

$$
(u_{\infty}') \text{ is bounded in } L^\infty(0, T; H^1_0(\Omega)) 
$$

$$
(u_{\infty}) \text{ is bounded in } L^2(0, T; H^1(\Omega)) 
$$

2.1.3 Limits of the Approximated Solutions

From the estimates (2.17)-(2.20) and (2.26)-(2.29), there exists a subsequence of $(u_{\infty})$, which we still denote by $(u_{\infty})$, such that:

$$
u_{\infty} \to u \text{ weakly star in } L^\infty(0, T; H^1_0(\Omega)) 
$$

$$
u_{\infty}' \to u' \text{ weakly star in } L^\infty(0, T; H^1_0(\Omega)) 
$$

$$
u_{\infty}' \to u' \text{ weakly in } L^2(0, T; H^1(\Omega)) 
$$

$$
u_{\infty}' \to 0 \text{ weakly in } L^\infty(0, T; L^2(\Omega)) 
$$

$$
u_{\infty} \to Ku \text{ weakly star in } L^\infty(0, T; L^2(\Omega)) 
$$

By (2.18), (2.19) and compactness arguments we conclude that there exists a subsequence of $(u_{\infty})$, which we still denote by $(u_{\infty})$, such that:

$$u_{\infty} \to u \text{ strongly in } L^2(0, t; L^2(\Omega)) = L^2(Q). 
$$

Thus,

$$u_{\infty} \to u \text{ almost everywhere in } Q. 
$$

whence, by (H.1) we have

$$F(u_{\infty}) \to F(u) \text{ almost everywhere in } Q 
$$

Since $K \in C^1([0, T; L^\infty(\Omega))]$, using (2.32) we obtain

$$
(Ku_{\infty}) \text{ is bounded in } L^2(Q) 
$$

Then,

$$
Ku_{\infty} \to Ku \text{ weakly in } L^2(Q) 
$$

Taking $w = u_{\infty}(t)$ in (2.13), integrating from 0 to $t$ and using (2.18), (2.19) and (2.37), we get

$$
\int_Q F(u_{\infty}(t))u_{\infty}(t)dxdt \leq C
$$

where $C$ is a positive constant.

By (2.36), (2.39) and Strauss's theorem (see Strauss [1]) it follows that

$$
F(u_{\infty}) \to F(u) \text{ weakly in } L^1(Q) 
$$
Multiplying (2.13) by \( \theta \in L^2(0, T) \), integrating from 0 to \( t \) and taking the limit as \( m \to \infty \) and \( \varepsilon \to 0 \), we obtain, by (2.30)-(2.34), (2.38) and (2.40):

\[
\left( \int_0^T K u \theta dt, \omega \right) + \left( \int_0^T -\Delta u \theta dt, w \right) + \left( \int_0^T -\Delta u \theta dt, w \right) + \left( \int_0^T F(u) \theta dt, w \right) = 0, \quad \forall w \in V_m.
\]

Since the \( V_m \) is dense in \( H^1_0(\Omega) \), the above equation is true for all \( w \in H^1_0(\Omega) \) and the proof of (2.11) is complete.

The initial conditions (2.12) are obtained from (2.30)-(2.32).

The uniqueness is trivial because \( F \) is Lipschitzian.

3. PROOF OF THEOREM 1

We first approximate \( u_0 \) by a sequence of bounded functions \( (u_{0j})_{j \in \mathbb{N}} \) in \( H^1_0(\Omega) \). In fact, let's consider

\[
\beta_j(s) = \begin{cases} 
 s & \text{if } |s| \leq j \\
 j & \text{if } s > j \\
 -j & \text{if } s < -j
\end{cases}
\]

it follows by Kinderlher-Stampacchia [8] that \( \beta_j(u_0) = u_{0j} \in H^1_0(\Omega) \), \( \forall j \in \mathbb{N} \), \( u_{0j} \to u_0 \) strongly in \( H^1_0(\Omega) \) and \( \|u_{0j}\| \leq \|u_0\| \).

Let \( (F_n)_{n \in \mathbb{N}} \) be a sequence of functions defined by:

\[
F_n(s) = \begin{cases} 
(-\eta)[G\left(s + \frac{1}{\eta}\right) - G(s)] & \text{if } -\eta \leq s \leq -\frac{1}{\eta} \\
(\eta)[G\left(s - \frac{1}{\eta}\right) - G(s)] & \text{if } \frac{1}{\eta} \leq s \leq \eta \\
\text{linear by parts} & \text{on } -\frac{1}{\eta} \leq s \leq \frac{1}{\eta} \text{ with } F_n(0) = 0 \\
\text{appropriated constants} & \text{for } |s| > \eta
\end{cases}
\]

where

\[
G(s) = \int_0^s F(\xi) d\xi.
\]

It follows, by Strauss [1], Cooper-Medeiros [7] that \( F_n \) is Lipschitzian, for each \( \eta \in \mathbb{N} \), \( sF_n(s) \geq 0 \) and \( F_n \to F \) uniformly on the bounded sets of \( \mathbb{R} \). If we consider \( G_n(s) = \int_0^s F_n(\xi) d\xi \) we get, \( G_n(0) = 0 \) and \( sG_n(s) \geq 0, \forall s \in \mathbb{R}, \forall \eta \in \mathbb{N} \).

Let \( \phi_{n\mu} \in D(\Omega) \) such that

\[
\phi_{n\mu} \to u_{0j} \text{ strongly in } H^1_0(\Omega) \text{ as } \mu \to \infty
\]  

It follows by Theorem 2 that there exists a unique function \( u_{\mu n} \) satisfying the conditions:
We now prove that $u_{\eta/n}$ converges to $u$ and $u$ is the solution of Theorem 1. Taking the inner product of (3.5) by $u_{\eta/n}$ and integrating from 0 to $T$, we have:

$$
(K, u_{\eta/n}^2) + \| u_{\eta/n} \|^2 + 2 \int_{\Omega} G_n(u_{\eta/n}) \, dx + 2 \int_0^T \| u_{\eta/n} \|^2 \, ds
$$

$$
\leq \| \phi_\eta \|^2 + 2 \int_{\Omega} G_n(\phi_\eta) \, dx + \int_0^T \| u_{\eta/n} \|^2 + C(\delta)(K, u_{\eta/n}^2) \, ds .
$$

(3.7)

Since $u_\eta$ is bounded in $\Omega$, fixing $\eta$, we obtain:

$$
F_\eta(u_{\eta/x}(x)) \to F(u_{\eta/x}(x)) \text{ uniformly in } \Omega \text{ as } \eta \to \infty ,
$$

(3.8)

and

$$
(G_\eta(u_{\eta/x}(x)) \to G(u_{\eta/x}(x)) \text{ uniformly in } \Omega \text{ as } \eta \to \infty .
$$

(3.9)

Moreover, $G(u_{\eta/x}) \to G(u_{\eta/x})$ a.e. in $\Omega$ and $G(u_{\eta/x}) \leq G(u_{\eta/x})$. Since $G(u_{\eta/x}) \in L^1(\Omega)$, by the Lebesgue's dominated convergence theorem we get

$$
\int_{\Omega} | G(u_{\eta/x}) - G(u_{\eta/x}) | \, dx \to 0 \text{ as } j \to \infty ,
$$

(3.11)

Thus, by (3.11) and (3.12), it follows that

$$
\int_{\Omega} G(u_{\eta/x}) \, dx \to \int_{\Omega} G(u_{\eta/x}) \, dx \text{ as } j \to \infty
$$

(3.13)

By (3.7), (3.9), (3.13) and Gronwall's inequality, we have

$$
(K, u_{\eta/x}^2) + \| u_{\eta/x} \|^2 + 2 \int_{\Omega} G_j(u_{\eta/x}) \, dx + (2 - C\delta) \int_0^T \| u_{\eta/x} \|^2 \, dx \leq C ,
$$

(3.14)

where $C$ is a positive constant independent of $\mu$, $j$ and $t$.

Then, there exists a subsequence of $(u_{\eta/j})_{\eta/x} \in \mathbb{W}$ which we denote by $(u_{\eta/j})_{\eta/x}$, and functions $u_j$ and $u$ such that
\[ |K^{1/2}u_j^j| \to |K^{1/2}u_j^j| \text{ weakly - star in } L^\infty(0,T;L^7(\Omega)) \]
\[ u_j \to u \text{ weakly - star in } L^\infty(0,T;H^1_0(\Omega)) \]
\[ u_j' \to u' \text{ weakly in } L^2(0,T;H^1_0(\Omega)) \]

as \( \mu \to \infty \), and

\[ |K^{1/2}u_j'| \to |K^{1/2}u'| \text{ weakly - star in } L^\infty(0,T;L^7(\Omega)) \]
\[ u_j \to u \text{ weakly - star in } L^2(0,T;H^1_0(\Omega)) \]
\[ u_j' \to u' \text{ weakly in } L^2(0,T;H^1_0(\Omega)) \]

as \( j \to \infty \).

Moreover, by (H.2) and \( K^{1/2}u_j^j \in L^\infty(0,T;L^2(\Omega)) \) if follows that:
\begin{equation}
Ku_j^j \in L^\infty(0,T;L^2(\Omega))
\end{equation}
and
\begin{equation}
Ku_j^j \to Ku_j^j \text{ weakly - star in } L^\infty(0,T;L^2(\Omega))
\end{equation}
as \( \mu \to \infty \), and

\begin{equation}
Ku_j^j \to Ku_j^j \text{ weakly - star in } L^\infty(0,T;L^2(\Omega))
\end{equation}
as \( j \to \infty \).

By (H.2), (H.3), (3.3) and (3.4) we get
\begin{equation}
(Ku_j') \in L^2(Q).
\end{equation}

(3.20)

So, by (3.18) and (3.19) we have that \( Ku_j^j \) is weakly continuous of \([0,T]\) in \( L^2(\Omega) \). Moreover, \((Ku_j^j)(T)\) is bounded in \( L^2(\Omega) \).

Multiplying (3.5) by \( u_j(t)\) and integrating from 0 to \( T \), we obtain
\begin{equation}
\int_0^T (F_j(u_j),u_j) dt \leq \int_0^T \|u_j\|^2 dt + \int_0^T \left| \left( \frac{\partial K}{\partial t} u_j, u_j' \right) \right| dt + \int_0^T \left| \left( \frac{\partial K}{\partial t} u_j, u_j' \right) \right| dt
\end{equation}
\[ + \int_0^T |a(u_j',u_j)| dt + \left| ((Ku_j^j)(T),u_j(T)) \right| + \left| ((Ku_j^j)(0),u_j(0)) \right|. \]

(3.21)

Using (H.2), (H.3) and a priori estimates, it follows that
\begin{equation}
\int_0^T F_j(u_j)u_j dx dt \leq C,
\end{equation}
(3.22)

\( C \) positive constant independent of \( \mu, j \) and \( t \).

Just as in Theorem 1, we prove that:
\begin{equation}
F_j(u_j) \to F(u_j) \text{ a.e. in } Q \text{ as } \mu \to \infty
\end{equation}
(3.23)

whence by (3.22), (3.23) and Strauss’s theorem (see Strauss [1]), we have
\begin{equation}
F_j(u_j) \to F_j(u_j) \text{ weakly in } L^1(Q) \text{ as } \mu \to \infty.
\end{equation}
(3.24)

Also, by (H.3) and (3.14) it follows that
\begin{equation}
(K'u_j^j) \text{ is bounded in } L^2(Q).
\end{equation}
(3.25)

So
\begin{equation}
K'u_j^j \to K'u_j^j \text{ weakly in } L^2(Q) \text{ as } j \to \infty
\end{equation}
(3.26)
and
\[ K'u_j \rightharpoonup K'u \text{ weakly in } L^2(\Omega) \text{ as } j \to \infty. \tag{3.27} \]

Multiplying (3.5) by \( w = \nu \theta \) with \( \nu \in H^1_0(\Omega) \) and \( \theta \in \mathcal{D}(0, T) \), integrating from 0 to \( T \), taking the limit as \( \mu \to \infty \), and using (3.15), (3.16), (3.18), (3.24) and (3.26) we get
\[ \frac{d}{dt}( Ku_j, \nu) - (K'u_j, \nu) + a(u_j, \nu) + a(u'_j, \nu) + (F(u_j), \nu) = 0 \quad \forall \nu \in H^1_0(\Omega) \text{ in } \mathcal{D}'(0, T). \tag{3.28} \]
\[ u_j(0) = u_{0j} \text{ and } (K'u_j)(0) = 0. \tag{3.29} \]

Moreover, by (3.24), it follows that:
\[ F_j(u_j) \rightharpoonup F(u) \text{ weakly in } L^1(\Omega). \tag{3.30} \]

Taking the limit in (3.28) as \( j \to \infty \) and using (3.16), (3.19), (3.27) and (3.30) we prove (2.1)-(2.5) in theorem 1.

It's not difficult to check that \( u(0) = u_0 \) and \((K'u')(0) = 0\).

REMARK. Replacing (H.2) by (H.2)' \( K \in C^1([0,T] ; L^\infty(\Omega)) \) with \( K(x, 0) \geq \alpha > 0 \),
\[ K(x, t) \geq 0, \quad (x, t) \in \Omega. \]

we get with the same arguments

THEOREM 3. Under hypotheses (H.1), (H.2)', (H.3) if \( G(s) = \int_0^s F(\xi)d\xi \) and \( u_0 \in H^1_0(\Omega) \), \( u_1 \in L^2(\Omega), G(u_0) \in L^1(\Omega) \), then there exists a function \( u : [0, T] \to L^2(\Omega) \) such that
\[ u \in L^\infty(0, T; H^1_0(\Omega)) \]
\[ u' \in L^2(0, T; H^1_0(\Omega)) \]
\[ \sqrt{K}u' \in L^\infty(0, T; L^2(\Omega)) \]
\[ K'u' \in L^2(0, T; H^1_0(\Omega)) \]
\[ Ku'' - \Delta u - \Delta u' + F(u) = 0 \text{ in the weak sense in } \Omega \]
\[ u(0) = u_0 \]
\[ u'(0) = u_1. \]

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REFERENCES


