SPACES OF COMPACT OPERATORS WHICH ARE M-IDEALS IN L(X, Y)

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ABSTRACT. Suppose X and Y are reflexive Banach spaces. If K(X, Y), the space of all compact linear operators from X to Y, is an M-ideal in L(X, Y), the space of all bounded linear operators from X to Y, then the second dual space K(X, Y)** of K(X, Y) is isometrically isomorphic to L(X, Y).

KEY WORDS AND PHRASES. Compact operators, M-ideal, dual space, projective tensor product.


1. INTRODUCTION

It is well known that if X and Y are reflexive Banach spaces one of which satisfies the approximation property then the second dual space K(X, Y)** of K(X, Y), the space of compact linear operators from X to Y, is isometrically isomorphic to L(X, Y), the space of bounded linear operators from X to Y (Diestel [1, p.17]). Harmand and Lima [2] proved that if X is a reflexive Banach space and K(X) is an M-ideal in L(X) then K(X)** is isometrically isomorphic to L(X).

The purpose of this paper is to generalize the result of Harmand and Lima to the case of K(X, Y) and L(X, Y) by modifying their proof. In Theorem 3.3 we will prove that if X and Y are reflexive Banach spaces and K(X, Y) is an M-ideal in L(X, Y) then K(X, Y)** is isometrically isomorphic to L(X, Y).

2. NOTATIONS AND PRELIMINARIES.

Let X and Y be Banach spaces. X \_Y means that X and Y are isometrically isomorphic. L(X, Y) (resp. K(X, Y)) will denote the space of all bounded linear operators (resp. compact linear operators) from X to Y. If X = Y, then we simply write L(X) (resp. K(X)). X* will denote the dual space of X and we will write \langle z, z* \rangle for the action of z* (X* on z (X instead of z*(z). B X will denote the closed unit ball of X.

A closed subspace J of a Banach space X is called an L-summand if there exists a projection P on X such that PX = J and \| z \| = \| Pz \| + \| z - Pz \| for every z in X. In this case we write X - J$1'I (I-P)X. A closed subspace of a Banach space X is called an M-ideal in X if J o, the annihilator of \in in X*, is an L-summand in X*.

Let Z be another Banach space and \in L(X, Z). We define Tu, \in X \_Y for u, \in X \_Y with \sum_{i=1}^{\infty} \| z_i \| \| y_i \| < \infty, the infimum being taken over all representations u = \sum_{i=1}^{\infty} z_i \_y_i. Moreover, we have \| u \| = \inf \sum_{i=1}^{\infty} \| z_i \| \| y_i \| < \infty, the infimum being taken over all representations u = \sum_{i=1}^{\infty} z_i \_y_i, z_i \in X, y_i \in Y and \sum_{i=1}^{\infty} \| z_i \| \| y_i \| < \infty. (Diestel and Uhl [3, p. 227]).

Let Z be another Banach space and T \in L(X, Z). We define Tu = \sum_{i=1}^{\infty} z_i \_T y_i, for u = \sum_{i=1}^{\infty} z_i \_y_i \in X \_Y. Then Tu \in X \_Z and \| Tu \| \leq \| T \| \| u \|. If u = \sum_{i=1}^{\infty} z_i^* \_x_i \in X^* \_X with \sum_{i=1}^{\infty} \| z_i^* \| \| x_i \| < \infty, the map u \rightarrow \text{tr}(u) = \sum_{i=1}^{\infty} < x_i, z_i^* > defines a bounded linear functional on X^* \_X with norm no larger than 1.
THEOREM 2.1 (Diestel and Uhl [3], Shatten [4]). Let $X$ and $Y$ be Banach spaces. The Banach space $L(X, Y^*)$ is isometrically isomorphic to $(Y \hat{\otimes} X)^*$ and under this identification $T \in L(X, Y^*)$ acts on $u \in Y \hat{\otimes} X$ by $\langle u, T \rangle = \text{tr}(Tu)$. 

THEOREM 2.2 (Feder and Sapher [5]). Let $X$ and $Y$ be Banach spaces. If either $X^{**}$ or $Y^*$ has the Radon-Nikodym Property, then map $V: Y^* \hat{\otimes} X^{**} \to K(X, Y)^*$ defined by $\langle T, V(u) \rangle = \text{tr}(T^{**}u)$ for $T \in K(X, Y)$ and $u \in Y^* \hat{\otimes} X^{**}$ is a quotient map.

3. SPACES OF COMPACT OPERATORS

Harmand and Lima [2] proved that if $K(X)$ is an $M$-ideal in $L(X)$ then there exists a net $(T_\alpha)$ in $B_K(X)$ such that

- $T_\alpha x \to x$ for all $x \in X$
- $T_\alpha x^* \to x^*$ for all $x^* \in X^*$
- $\|T_\alpha - I\| \to 1$.

In the case of $K(X, Y)$ and $L(X, Y)$, we have the following analogue which also plays a key role in the proof of our main result (Theorem 3.3).

THEOREM 3.1. If $X$ and $Y$ are Banach spaces and $K(X, Y)$ is an $M$-ideal in $L(X, Y)$, then for each $T$ in $B L(X, Y)$ there is a net $(T_\alpha)$ in $B_K(X, Y)$ such that

- $T_\alpha x \to x$ for all $x \in X$
- $T_\alpha y^{**} \to T^{**}y^{**}$ for all $y^{**} \in Y^{**}$

PROOF. Suppose $K(X, Y)$ is an $M$-ideal in $L(X, Y)$. Then we can write $L(X, Y)^* = (K(X, Y))^* \oplus J$ for some subspace $J$ of $L(X, Y)^*$. The map $\sigma \to \sigma + K(X, Y)^0$ defines an isometry from $J$ onto $L(X, Y)^*/K(X, Y)^0$ and the map $\sigma + K(X, Y)^0 \to K(X, Y)$ defines an isometry from $J$ onto $K(X, Y)^0$ (Rudin [6, p.91]). Hence the map $\sigma \to \sigma + K(X, Y)$ gives an isometry from $J$ onto $K(X, Y)^0$. Let $Q$ be the projection on $L(X, Y)^*$ with the range $J$. Then if $\sigma \in L(X, Y)^*$ is in the range of $Q$ if and only if the restriction of $\sigma$ to $K(X, Y)$ has the same norm as $\sigma$. If $T \in L(X, Y) \leq L(X, Y)^*$ with $\|T\| \leq 1$, then for $\sigma \in K(X, Y)^0$ we have $(Q^*T)\sigma = TQ(\sigma) = 0$ thus $Q^*T \in K(X, Y)^0 \leq J^* = K(X, Y)^0$. Since $Q^*T \in K(X, Y)^0$ and $\|Q^*T\| \leq 1$, by the Goldstein's theorem there is a net $(T_\alpha)$ in $B_K(X, Y)$ such that

$T_\alpha \to Q^*T$ in the weak*-topology on $J^{**} = K(X, Y)^{**}$. We claim that $T_\alpha x \to T x$ for all $x \in X$ and $T_\alpha y^{**} \to T^{**}y^{**}$ for all $y^{**} \in Y^{**}$. For $z^{**} \in X^{**}$ and $y^* \in Y^*$, define

$s_{z^{**}} \otimes y^* \in L(X, Y)^*$ by

$\langle A, s_{z^{**}} \otimes y^* \rangle = \langle A^{**} y^*, z^{**} \rangle$.

Then we can easily see that $s_{z^{**}} \otimes y^* \in J = K(X, Y)^*$ and hence

$T_\alpha y^{**} \to T^{**}y^{**}$ for all $y^{**} \in Y^{**}$. Similarly, for $y^* \in Y^*$ and $x \in X$ the functional $\psi_{y^*} \otimes x$ on $L(X, Y)$ defined by $\langle A, \psi_{y^*} \otimes x \rangle = \langle Ax, y^* \rangle$ for $A \in L(X, Y)$ is in the range of $Q$ and hence $T_\alpha x \to T x$ for all $x \in X$.

The following proposition is essentially due to Harmand and Lima [2] who treated a special case $X = Y$.

PROPOSITION 3.2. Let $X$ and $Y$ be Banach spaces and $V$ the map defined in Theorem 2.2. If $K(X, Y)$ is an $M$-ideal in $L(X, Y)$, then $T^{**} \in (ker V)^0$ for every $T \in L(X, Y)$.

PROOF. Recall that by Theorem 2.1 we have $(Y^* \hat{\otimes} X^{**})^* \simeq L(X^{**}, Y^{**})$ and under this identification $S \in L(X^{**}, Y^{**})$ acts on $u \in Y^* \hat{\otimes} X^{**}$ by $\langle u, S \rangle = \text{tr}(Su)$. Let $T \in L(X, Y)$, $\|T\| \leq 1$. By Theorem 3.1 there is a net $(T_\alpha)$ in $B_K(X, Y)$ such that $T_\alpha^{**} y^{**} \to T^{**} y^{**}$ for all $y^{**} \in Y^{**}$. Let $u = \bigoplus_{i=1}^{\infty} y_i^* \otimes z_i^{**} \in ker V$ with $\bigoplus_{i=1}^{\infty} y_i^* \|z_i^{**}\| < \infty$. We may assume that $\|z_i^{**}\| \leq 1$ for all $i$. 

\[ \| y^*_i \| \to 0. \] Then we get

\[
0 = \langle T_{\alpha}, V(u) \rangle = \text{tr}(T_{\alpha}^* u) = \sum_{i=1}^{\infty} \langle y^*_i, T_{\alpha}^* y_i^* \rangle = \sum_{i=1}^{\infty} \langle T_{\alpha}^* y_i^*, z_i^* \rangle = \sum_{i=1}^{\infty} \langle T_{\alpha}^* y_i^*, z_i^* \rangle = \text{tr}(T_{\alpha}^* u) = \langle u, T_{\alpha}^* \rangle.
\]

Thus \( T_{\alpha}^* \in (\ker V)^{0}. \)

**THEOREM 3.3.** If \( X \) and \( Y \) are reflexive Banach spaces and \( K(X,Y) \) is an \( M \)-ideal in \( L(X,Y) \) then \( K(X,Y)^{**} \) is isometrically isomorphic to \( L(X,Y) \).

**PROOF.** Since \( X \) and \( Y \) are reflexive, \( X \) and \( Y^* \) have the Radon-Nikodym property and hence by Theorem 2.2 the map \( V: Y^* \otimes X^{**} \rightarrow K(X,Y)^* \) defined by

\[
\langle T, V(u) \rangle = \text{tr}(T^* u) \quad \text{for} \quad u \in Y^* \otimes X^{**}, \quad T \in K(X,Y)
\]

is a quotient map. Thus \( V^*: K(X,Y)^{**} \rightarrow (Y^* \otimes X^{**})^* \) is an isometry with the range \((\ker V)^{0}\) and hence we have

\[
K(X,Y)^{**} \simeq (\ker V)^{0} \\
\subseteq (Y^* \otimes X^{**})^* \\
\simeq L(X^{**}, Y^{**}) \\
= L(X,Y).
\]

Since \( X \) and \( Y \) are reflexive, \( T = T_{\alpha}^* \) for all \( T \in L(X,Y) \) and by Proposition 3.2 \((Y^* \otimes X^{**})^* \subseteq (\ker V)^{0}\).

Thus \( K(X,Y)^{**} \simeq L(X,Y) \).

Recall that for \( 1 \leq p \leq \infty \) the \( l_p \)-sum \( (\Sigma X_n)_p \) of a sequence of \( (X_n) \) of Banach spaces is the Banach space of all sequences \( (x_n) \) with \( x_n \in X_n \) and with the norm \( \| (x_n) \| = (\Sigma \| x_n \|_p)^{1/p} < \infty \).

**COROLLARY 3.4.** Suppose \( X \) and \( Y \) are closed subspaces of \( (\Sigma X_n)_p \) and \( (\Sigma Y_n)_q \) \((1 < p \leq q \leq \infty, \ dim X_n < \infty, \ dim Y_n < \infty, \) respectively. If \( K(X,Y) \) is dense in \( L(X,Y) \) in the strong operator topology, then \( K(X,Y)^{**} \) is dense in \( L(X,Y) \).

**PROOF.** If \( X \) and \( Y \) are reflexive and \( K(X,Y) \) is an \( M \)-ideal in \( L(X,Y) \) (Cho [7]).

**REMARK.** If \( X \) and \( Y \) are as in Corollary 3.4 and either \( X \) or \( Y \) satisfies the compact approximation property, then \( K(X,Y) \) is dense in \( L(X,Y) \) in the strong operator topology [7] and hence \( K(X,Y)^{**} \) is dense in \( L(X,Y) \).

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**REFERENCES**


