ABSTRACT. A weak right $H^*$-algebra is a Banach algebra $A$ which is a Hilbert space and which has a dense subset $D_r$ with the property that for each $x$ in $D_r$ there exists $x^r$ such that $(yx, z) = (y, zz^r)$ for all $y, z$ in $A$. It is shown that a proper (each $x^r$ is unique) weak right $H^*$-algebra is semi-simple. Also there is an example of weak right $H^*$-algebra which is not a left $H^*$-algebra.

KEY WORDS AND PHRASES. Hilbert algebra, $H^*$-algebra, weak right $H^*$-algebra, weak left $H^*$-algebra, complemented algebra, right complemented algebra, left complemented algebra.


1. INTRODUCTION.

Assumption of semi-simplicity plays an important role in the theory of complemented algebras. It was noted in the author's last paper (Saworotnow [1]) that it is rather difficult to deduct semi-simplicity from axioms of a (proper) weak right $H^*$-algebra. However, there is a different story for the case of a two-sided (weak) $H^*$-algebra. Here it is not too difficult to show that each closed two-sided ideal has an idempotent which, in turn, implies semi-simplicity. But it was established in Saworotnow [1] that each proper weak right $H^*$-algebra is also a weak left $H^*$-algebra. It follows that each proper right $H^*$-algebra is semi-simple (Theorem 2 below). This is the central result of this paper. We included also important consequences of it and an example of an algebra which is a right $H^*$-algebra but not a left $H^*$-algebra. The algebra in the example is also an example of a weak right $H^*$-algebra which is not a weak left $H^*$-algebra.
a member $a^r$ of $D_r$ such that $(xa, y) = (x, ya^r)$ for all $x, y \in A$; $a^r$ is called the right adjoint of $a$. It is said to be proper if $a^r$ is unique for every $a$ in $D_r$; this is equivalent to the condition that the right annihilator $r(A) = \{x \in A : Ax = 0\}$ of $A$ consists of zero alone ($A$ is proper if and only if $r(A) = \{0\}$).

We define weak left $H^*$-algebra in a similar way. Weak two-sided $H^*$-algebra is a weak right $H^*$-algebra which is also a (weak) left $H^*$-algebra.

**THEOREM 1.** Every weak right $H^*$-algebra is a right complemented algebra (Saworotnow [2]), i.e., the orthogonal complement $R^p$ of any right ideal $R$ in $A$ is also a right ideal.

**PROOF.** If $x \in R$ and $a \in A$, then $(xa, y) = \lim(xa_n, y) = \lim(x, ya_n) = 0$ for some sequence $(a_n) \subset D_r$ converging to $a$ and each $y \in R$. This implies that $R^p$ is also a right ideal.

**PROPOSITION 1.** The orthogonal complement $P'$ of each two-sided $I$ in a weak right $H^*$-algebra $A$ is again a weak right $H^*$-algebra. (Note that we do not allege $I$ itself to be a weak right $H^*$-algebra.)

**PROOF.** First note that $P' \subseteq P \cap I = \{0\}$, i.e., $xy = 0$ for all $x \in P, y \in I$.

Now consider $a \in P$ and let $\epsilon > 0$ be arbitrary. Take $b \in D_r$ so that $\|a - b\| < \epsilon$ and write $b = b_1 + b_2$, $b^r = c_1 + c_2$ with $b_1, c_1 \in P$ and $b_2, c_2 \in I$. Then $\|a - b_1\| < \epsilon$ and we have for each $x, y \in P$:

$$(zb, y) = (xb + xb_2, y) = (xb, y) = (x, yb^r) = (x, yc_1 + yc_2) = (x, yc_1),$$

which simply means that $c_1$ is a right adjoint of $b_1$. Thus: every neighborhood of $a$ contains a vector having a right adjoint.

**PROPOSITION 2.** Each closed two-sided ideal $I$ in a proper weak right $H^*$-algebra $A$ is a proper weak right $H^*$-algebra. In fact, it is also a weak left $H^*$-algebra.

**PROOF.** It was shown in Saworotnow [1] that $A$ is also a proper weak left $H^*$-algebra. This means that $P'$ is also a left ideal (we can use here the proof of Theorem 1 above). Thus: $I$ is the orthogonal complement of a two-sided ideal. Proposition 2 now follows from Proposition 1 ($I$ is the orthogonal complement of the two-sided ideal $P'$); the fact that $I$ is proper is also easy to establish.

3. **MAIN THEOREM.**

Now we can prove our main result.

**THEOREM 2.** Every proper weak right $H^*$-algebra $A$ is semi-simple.

**PROOF.** Proposition 2 implies that the radical (Jacobson [3]) $R$ of $A$ is a right $H^*$-algebra. Hence it contains a non-zero vector $a$ having a (unique) right adjoint $a^r \neq 0$. Then $aa^r \neq 0$ (otherwise $\|xa\|^2 = (x, xaa^r) = 0$ for each $x \in A$) and as in 27A of Loomis [4] one can show that, for some scalar $\lambda$, the sequence $\{\lambda aa^r\}_{n=1}^{\infty}$ converges to some idempotent $e \in R$. This is impossible since every member of $R$ is a generalized nilpotent (Theorem 16, page 309 in Jacobson [3]).

An important consequence of this theorem is the fact that we can now apply to the algebra $A$ the theory of complemented algebras developed in Saworotnow [2] and Saworotnow [5] (more
specifically: Theorem 1 in Saworotnow [2] and Theorem 3 in Saworotnow [5]. We summarize it as follows:

THEOREM 3. Every proper weak right $H^*$-algebra is a direct sum of simple weak right $H^*$-algebras, each of which is a semi-simple.

THEOREM 4. For each proper simple weak right $H^*$-algebra $A$ there is a Hilbert space $H$ and a positive self-adjoint norm-increasing operator $\alpha$ on $H$ such that $A$ is isomorphic and isometric to the algebra of all Hilbert Schmidt operators $a$ on $H$ such that $aa^* \alpha$ is also of Hilbert Schmidt type.

This means that each simple proper weak right (as well as left) $H^*$-algebra is of the type described in the Example on page 56 of Saworotnow [5].

4. AN EXAMPLE.

To conclude the paper, we give an example of a right $H^*$-algebra which is not a weak left $H^*$-algebra. This example shows that our assumption of an algebra to be proper is rather essential.

EXAMPLE. Let $A$ be the algebra of all $2 \times 2$ matrices and let

\[ e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \]  

Consider the subalgebra $A_0$ of $A$ generated by $e_1$ and $e_{21}$, $A_0 = \{ \lambda e_1 + \mu e_{21} : \lambda, \mu \text{ complex} \}$. Then $A_0$ is a right (as well as a weak right) $H^*$-algebra (note that $\lambda e_1$ is a right adjoint of $\lambda e_1 + \mu e_{21}$).

But $A_0$ could not be a left weak $H^*$-algebra since the orthogonal complement $L^\perp = \{ e_1 \}$ of the left ideal $L = \{ e_{21} \}$ is not a left ideal (here $\{ x \}$ denotes the 1-dimensional subspace of $A$ generated by $x$). Note that $r(A_0) = (0)$ and $\ell(A_0) = L$ (here $\ell(A_0)$ denotes the left annihilator of $A_0$, $\ell(A_0) = \{ x : xA = 0 \}$).

REFERENCES


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