ON POLYNOMIAL EP, MATRICES

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ABSTRACT. This paper gives a characterization of EP_r-\lambda-matrices. Necessary and sufficient conditions are determined for (i) the Moore-Penrose inverse of an EP_r-\lambda-matrix to be an EP_r-\lambda-matrix and (ii) Moore-Penrose inverse of the product of EP_r-\lambda-matrices to be an EP_r-\lambda-matrix. Further, a condition for the generalized inverse of the product of \lambda-matrices to be a \lambda-matrix is determined.


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1. INTRODUCTION

Let \( \mathbb{P}_X \) be the set of all \( m \times n \) matrices whose elements are polynomials in \( \lambda \) over an arbitrary field \( F \) with an involutory automorphism \( \sigma : a \mapsto \bar{a} \) for \( a \in F \). The elements of \( \mathbb{P}_X \) are called \( \lambda \)-matrices. For \( A(\lambda) = (a_{ij}(\lambda)) \in \mathbb{P}_X \), \( A^*(\lambda) = (\bar{a}^*_{ij}(\lambda)) \). Let \( \mathbb{P}^{\text{reg}}_X \) be the set of all \( m \times n \) matrices whose elements are rational functions of the form \( f(\lambda)/g(\lambda) \) where \( f(\lambda), g(\lambda) \neq 0 \) are polynomials in \( \lambda \). For simplicity, let us denote \( A(\lambda) \) by \( A \) itself.

The rank of \( A \in \mathbb{P}^{\text{reg}}_X \) is defined to be the order of its largest minor that is not equal to the zero polynomial ([2], p. 259). \( A \in \mathbb{P}^{\text{reg}}_X \) is said to be a unimodular \( \lambda \)-matrix (or) invertible in \( \mathbb{P}^{\text{reg}}_X \) if the determinant of \( A(\lambda) \), that is, \( \det A(\lambda) \) is a nonzero constant. \( A \in \mathbb{P}^{\text{reg}}_X \) is said to be a regular \( \lambda \)-matrix if and only if it is of rank \( n \) ([2], p. 259), that is, if and only if the kernel of \( A \) contains only the zero element. \( A \in \mathbb{P}^{\text{reg}}_X \) is said to be EP_r over the field \( \mathbb{F}(\lambda) \) if \( \text{rk}(A) = r \) and \( \text{R}(A) = \text{R}(A^*) \) where \( \text{R}(A) \) and \( \text{rk}(A) \) denote the range space of \( A \) and rank of \( A \) respectively [4]. We have \{ unimodular \( \lambda \)-matrices \} \( \subset \) \{ regular \( \lambda \)-matrices \} \( \subset \) \{ EP_r-\lambda \)-matrices \}.

Throughout this paper, let \( A \in \mathbb{P}^{\text{reg}}_X \). Let \( 1 \) be identity element of \( F \). The Moore-Penrose inverse of \( A \), denoted by \( A^+ \) is the unique solution of the following set of equations:

\[
AXA = A \ (1.1); \quad XAX = X \ (1.2); \quad (AX)^* = AX \ (1.3); \quad (XA)^* = XA \ (1.4)
\]

\( A^+ \) exists and \( A \in \mathbb{P}^{\text{reg}}_X \) if and only if \( \text{rk}(AA^*) = \text{rk}(A^*A) = \text{rk}(A) \) [7]. When \( A^+ \) exists, \( A \) is EP_r over \( \mathbb{F}(\lambda) \) \( \iff \) \( AA^+ = A^+A \). For \( A \in \mathbb{P}^{\text{reg}}_X \), a generalized inverse (or) \{1\} inverse is defined as a solution of the polynomial matrix equation (1.1) and a reflexive generalized inverse (or) \{2\} inverse is defined as a solution of the equations (1.1) and (1.2) and they belong to \( \mathbb{P}^{\text{reg}}_X \). The purpose of this paper is to give a characterization of an EP_r-\lambda-matrix. Some results on EP_r-\lambda-matrices having the same range space are obtained. As an application necessary and sufficient conditions are derived for \( (AB)^+ \) to be an EP_r-\lambda-matrix whenever \( A \) and \( B \) are EP_r-\lambda-matrices.
2. CHARACTERIZATION OF AN EP-λ-MATRIX

**THEOREM 1.** Let $A \in \mathcal{M}_n(F)$ be an EP-λ-matrix over the field $F(\lambda)$. Then there exist $n \times n$ regular λ-matrices $P$ and $Q$ such that

$$P^{-1}AP = E \lambda^T$$

where $E$ is a $r \times r$ regular λ-matrix.

**PROOF.** By the Smith's canonical form, $A = D \lambda^T$ where $P$ and $Q$ are unimodular-λ-matrices of order $n$ and $D$ is a $r \times r$ regular diagonal λ-matrix. Any inverse of $A$ is given by $A^{-1} = Q_1 D^{-1} P_1$ where $Q_1$ and $P_1$ are arbitrary conformable matrices over $F(\lambda)$. $A$ is EP over the field $F(\lambda)$.

$$\Rightarrow R(A) = R(A^*)$$

$$\Rightarrow A = AA^*(A^*)$$

(By Theorem 1[3])

$$\Rightarrow \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Partitioning conformably, let $Q = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}$$

Therefore $Q = \begin{bmatrix} T_1 & 0 \\ T_3 & T_4 \end{bmatrix}$

Hence $A = P \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ T_3 & T_4 \end{bmatrix} \begin{bmatrix} 0 & P^* \\ 0 & 0 \end{bmatrix} = P \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$

where $E = D T_1$ is a $r \times r$ regular λ-matrix.

Conversely, let $P = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$ where $E$ is a $r \times r$ regular λ-matrix.

Since $E$ is regular, $E$ is EP over $F(\lambda)$.

$$\Rightarrow R(E) = R(E^*)$$

$$\Rightarrow R(PA^*) = R(PA^*P)$$

$$\Rightarrow A = EP^{-1}$$

If $A \in \mathcal{M}_n(F(\lambda))$ and is EP over the field $F(\lambda)$ then we can find $n \times n$ regular rational λ-matrices $H$ and $K$ such that $A = HA = AK$. In general the above $H$ and $K$ need not be unimodular λ-matrices. For example, consider $A = \begin{bmatrix} 1 & \lambda \\ \lambda & 2 \end{bmatrix}$. A is
EP, being a regular $\lambda$-matrix. If $A^* = HA$ then $H = A^* A^{-1}$; If $A^* = AK$ then $K = A^{-1} A^*$. Here $H = \begin{bmatrix} 1 & -1/\lambda \\ 0 & 1/\lambda \end{bmatrix}$ and $K = \begin{bmatrix} 0 & -\lambda \\ \lambda & 1 \end{bmatrix}$ are not $\lambda$-matrices.

The following theorem gives a necessary condition for $H$ and $K$ to be unimodular $\lambda$-matrices.

**Theorem 2.** If $A$ is an $n \times n$ EP-$\lambda$-matrix and $A$ has a $\lambda$-matrix (1) inverse then there exist $n \times n$ unimodular $\lambda$-matrices $H$ and $K$ such that $A H = A K$.

**Proof.** Let $A$ be an $n \times n$ EP-$\lambda$-matrix. By Theorem 1, there exists an $n \times n$ unimodular $\lambda$-matrix $P$ such that $P A P = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$ where $E$ is a $r \times r$ regular $\lambda$-matrix. Since $A$ has a $\lambda$-matrix (1) inverse, $E^{-1}$ is also a $\lambda$-matrix.

Now

$$A = P^{-1} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} P^{-1}$$

Therefore

$$A^* = P^{-1} \begin{bmatrix} E^* & 0 \\ 0 & 0 \end{bmatrix} P^{-1}$$

$$= P^{-1} \begin{bmatrix} E^* E^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}$$

$$= H A$$

where $H = P^{-1} \begin{bmatrix} E^* & 0 \\ 0 & 0 \end{bmatrix} P$ is an $n \times n$ unimodular $\lambda$-matrix. Similarly we can write $A^* = AK$ where $K = P^{-1} \begin{bmatrix} E^* & 0 \\ 0 & 0 \end{bmatrix} P$ is an $n \times n$ unimodular $\lambda$-matrix.

**Remark 1.** The converse of Theorem 2 need not hold. For example, consider $A = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}$. Since $A^* = A$, $H = K = I_2$. $A$ is an EP-$\lambda$-matrix. However, $A$ has no $\lambda$-matrix (1) inverse.

**3. Moore-Penrose Inverse of an EP-$\lambda$-Matrix**

The following theorem gives a set of necessary and sufficient conditions for the existence of the $\lambda$-matrix Moore-Penrose inverse of a given $\lambda$-matrix.

**Theorem 3.** For $A \in \mathbb{F}_r \times \mathbb{R}_r$, the following statements are equivalent.

i) $A$ is EP-$\lambda$-matrix $\lambda$-matrix and $A A$ has a $\lambda$-matrix (1) inverse.

ii) There exists an unimodular $\lambda$-matrix $U$ with $A = U \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} U^*$

where $D$ is a $r \times r$ unimodular $\lambda$-matrix and $U^* U$ is a diagonal block matrix.

iii) $A = GLG$ where $L$ and $G$ are $r \times r$ unimodular $\lambda$-matrices and $G$ is a $\lambda$-matrix.

iv) $A^*$ is a $\lambda$-matrix and EP-$\lambda$.

v) There exists a symmetric idempotent $\lambda$-matrix $E$, $(E^2 = E = E^*)$ such that

$$AE = EA \quad \text{and} \quad R(A) = R(E).$$

**Proof.** (i) $\Rightarrow$ (ii) Since $A$ is an EP-$\lambda$-matrix over the field $\mathbb{F}(\lambda)$ and $\text{rk}(A) = \text{rk}(A^2)$, $A^*$ exists, by Theorem 2.3 of [5]. By Theorem 4 in [6], $A A$ has a $\lambda$-matrix (1) inverse implies that there exists an unimodular $\lambda$-matrix $P$ with $P P^* = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$ where $P_1$ is a symmetric $r \times r$ unimodular $\lambda$-matrix such that
PA = \[\begin{bmatrix} W \\ 0 \end{bmatrix}\] where \(W\) is a \(\text{rxn}\), \(\lambda\)-matrix of rank \(r\). Hence by Theorem 2 in [6], 
\(AA^*\) is a \(\lambda\)-matrix and \(PAA^*P^* = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix}\). Since \(A\) is \(\text{EP}\), \(AA^* = A^*A\) and 
\(A = AA^*A = A(AA^*)\). Therefore 
\(A = P^{-1} \begin{bmatrix} W \\ 0 \end{bmatrix} P^*\) where 
\(P^{-1} \begin{bmatrix} H & 0 \end{bmatrix} P^*\)

\(H\) consists of the first \(r\) columns of \(P^*\), thus \(H\) is a \(n \times r\), \(\lambda\)-matrix of rank \(r\).

Now \(A = P^{-1} \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} P^* = U \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} U^*\) where \(U = P^{-1}\) and \(D = WH\) is a \(\text{rxr}\) regular \(\lambda\)-matrix. Since \(AA^*\) has a \(\lambda\)-matrix \(\{1\}\) inverse and \(P\) is an \(\text{unimodular} \lambda\)-matrix, \(PAA^*P^* = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}\) has a \(\lambda\)-matrix \(\{1\}\) inverse. Therefore by Theorem 1 in [6], \(D^{-1}D\) is an \(\text{unimodular} \lambda\)-matrix which implies \(D\) is an \(\text{unimodular} \lambda\)-matrix. Hence \(A = U \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} U^*\) where \(D\) is a \(\text{rxr} \text{unimodular} \lambda\)-matrix and \(U^*U\) is a diagonal block \(\lambda\)-matrix.

Thus (ii) holds.

(ii) \(\Rightarrow\) (iii)

Let us partition \(U\) as \(U = \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix}\) where \(U_1\) is a \(\text{rxr} \lambda\)-matrix. Then 
\(A = \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1^* & U_3^* \\ U_2^* & U_4^* \end{bmatrix} = D \begin{bmatrix} U_1^* & U_3^* \\ U_2^* & U_4^* \end{bmatrix} = GLG^*\)

where \(L = D\) and \(G = \begin{bmatrix} U_1 \\ U_3 \end{bmatrix}\) are \(\lambda\)-matrices.

Since \(U^*U\) is a diagonal block \(\lambda\)-matrix, \(G^*G = U_1^*U_1 + U_3^*U_3\) and \(L\) are \(\text{rxr} \text{unimodular} \lambda\)-matrices. Thus (iii) holds.

(iii) \(\Rightarrow\) (iv)

Since \(A = GLG^*\), \(L\) and \(G^*\) are \(\text{unimodular} \lambda\)-matrices. One can verify that 
\(A^* = G(G^*)^{-1}L^{-1}(G^*)^{-1}G\).

Now \(AA^* = GLG^*G(G^*)^{-1}L^{-1}(G^*)^{-1}G^* = G(G^*)^{-1}G = A^*A\) implies that \(A^*\) is \(\text{EP}\). Since \(L\) and \(G^*\) are \(\text{unimodular}\), \(L^{-1}\) and \((G^*)^{-1}\) are \(\lambda\)-matrices, and \(G\) is a \(\lambda\)-matrix. Therefore \(A^*\) is a \(\lambda\)-matrix. Thus (iv) holds.

(iv) \(\Rightarrow\) (v)

Proof is analogous to that of (ii) \(\Rightarrow\) (iii) of Theorem 2.3 [5].

(v) \(\Rightarrow\) (i)

Since \(E\) is a symmetric idempotent \(\lambda\)-matrix with \(R(A) = R(E)\) and \(AE = EA\), by Theorem 2.3 in [5] we have \(A\) is \(\text{EP}\) and \(\text{rk}(A) = \text{rk}(A^2) \Rightarrow A^*\) exists. Since \(E^* = E\) and \(R(A) = R(E) \Rightarrow AA^* = EE^* = E\). Now \(AE = EA = (AA^*)A = A\). Let \(e_j\) and \(a_j\) denote the \(j\)-th columns of \(E\) and \(A\) respectively. Then 
\(AE = A \Rightarrow Ae_j = a_j\), since \(e_j\) is a \(\lambda\)-matrix, the equation \(Ax = a_j\) where \(a_j\) is a \(\lambda\)-matrix, has a \(\lambda\)-matrix solution. Hence by Theorem 1 in [6] it follows that \(A\) has a \(\lambda\)-matrix \(\{1\}\) inverse. Further \(AA^* = E\) is also a \(\lambda\)-matrix. Hence by Theorem 4 in [6] we see that \(A^*A\) has a \(\lambda\)-matrix \(\{1\}\) inverse. Thus (i) holds.

Hence the theorem.
REMARK 2. The condition (i) in Theorem 3 cannot be weakened which can be seen by the following examples.

EXAMPLE 1. Consider the matrix $A = \begin{bmatrix} \lambda & \lambda \\ 2\lambda^2 & 2\lambda^2 \\ \lambda & \lambda \end{bmatrix}$. A is EP$_1$ and $\text{rk}(A) = \text{rk}(A^2) = 1$. $A^* A = \begin{bmatrix} 2\lambda^2 & 2\lambda^2 \\ 2\lambda^2 & 2\lambda^2 \end{bmatrix}$ has no $\lambda$-matrix $\{1\}$ inverse (since the invariant polynomial of $A^* A$ is $\lambda^2$ which is not the identity of $F$). For this $A$, $A^+ = \frac{1}{4\lambda} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is not a $\lambda$-matrix. Thus the theorem fails.

EXAMPLE 2. Consider the matrix $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ over GF(5). A is EP$_1$. Since $\text{rk}(A) \neq \text{rk}(A^2)$, $A A^*$ has a $X$-matrix $\{1\}$ inverse (since any conformable $\lambda$-matrix is a $\lambda$-matrix $\{1\}$ inverse). For this $A$, $A^*$ does not exist. Thus the theorem fails.

REMARKS 3. From Theorem 3, it is clear that if $E$ is a symmetric idempotent $\lambda$-matrix, and $A$ is a $\lambda$-matrix such that $R(E) = R(A)$ then $A$ is EP $\iff$ $A E = E A \iff A^*$ is a $\lambda$-matrix and EP.

We can show that the set of all EP$_\lambda$-matrices with common range space as that of given symmetric idempotent $\lambda$-matrix forms a group, analogous to that of Theorem 2.1 in [5].

COROLLARY 1. Let $E = E^* \in F_{\lambda}^{P\times q}$. Then $H(E) = \{A \in F_{\lambda}^{P\times q}: A \text{ is EP}_\lambda \text{ over } F(\lambda) \text{ and } R(A) = R(E)\}$ is a maximal subgroup of $F_{\lambda}^{P\times q}$ containing $E$ as identity.

PROOF. This can be proved similar to that of Theorem 2.1 of [5] by applying Theorem 3.

4. APPLICATION

In general, if $A$ and $B$ are $\lambda$-matrices, having $\lambda$-matrix $\{1\}$ inverses, it is not necessary that $A B$ has a $\lambda$-matrix $\{1\}$ inverse.

EXAMPLE 3. Consider $A = \begin{bmatrix} 1 & \lambda \\ \lambda & 2 \lambda \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 2\lambda & 0 \end{bmatrix}$. Here $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is one of the $\lambda$-matrix $\{1\}$ inverse for both $A$ and $B$. But $A B = \begin{bmatrix} 1 + 2\lambda^2 & 0 \\ \lambda + 2\lambda^2 & 0 \end{bmatrix}$. Since the invariant polynomial of $A B$ is $1 + 2\lambda^2 \neq 1$, $A B$ has no $\lambda$-matrix $\{1\}$ inverse.

The following theorem leads to the existence of $\lambda$-matrix $\{1\}$ inverse of the product $A B$.

THEOREM 4. Let $A, B \in F_{\lambda}^{P\times q}$. If $A^2 = A$ and $B$ has $\lambda$-matrix $\{1\}$ inverse and $R(A) \subseteq R(B)$ then $A B$ has a $\lambda$-matrix $\{1\}$ inverse.

PROOF. Suppose $A B x = b$, where $b$ is a $\lambda$-matrix, is a consistent system. Then $b \in R(AB) \subseteq R(A) \subseteq R(B)$ and therefore $B z_0 = b$. Since $B$ has a $\lambda$-matrix $\{1\}$ inverse, by Theorem 1 in [6] we get $z_0$ is a $\lambda$-matrix. Since $A$ is idempotent, so in particular $A$ is a $\{1\}$ inverse of $A$ and $b \in R(A)$, we have $A b = b$. Now $A B z_0 = A A B = A b = b$. Thus $A B x = b$ has a $\lambda$-matrix solution. Hence by Theorem 1 in [6], $A B$ has a $\lambda$-matrix $\{1\}$ inverse. Hence the theorem.

The converse of Theorem 4 need not be true which can be seen by the following example.

EXAMPLE 4. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 1 \\ \lambda & \lambda \end{bmatrix}$; $A B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Here $A^2 = A$ and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is a $\lambda$-matrix $\{1\}$ inverse for both $A B$ and $B$. However
R(A) \parallel R(B). Hence the converse is not true.

Next we shall discuss the necessary and sufficient condition for the Moore-Penrose inverse of the product of EP\_λ-matrices to be an EP\_λ-matrix.

**THEOREM 5.** Let A and B be EP\_λ-matrices. Then A\_+ A has a λ-matrix \{1\} inverse, \text{rk}(A) = \text{rk}(A^2) and R(A) = R(B) if and only if AB is EP\_λ and (AB)\_+ = B\_+ A\_+ is a λ-matrix.

**PROOF.** Since A and B are EP\_λ with R(A) = R(B) and \text{rk}(A) = \text{rk}(A^2), by a Theorem of Katz [1], AB is EP\_λ. Since A is an EP\_λ-matrix, \text{rk}(A) = \text{rk}(A^2) and A\_+ A has a λ-matrix \{1\} inverse, by Theorem 3, A\_+ is a λ-matrix and there exists a symmetric idempotent λ-matrix E such that R(A) = R(E). Hence AA\_+ = AA\_+ = E. Since A and B are EP\_λ and R(A) = R(B), we have AA\_+ = BB\_+ = E = A\_+ A = B\_+ B. Therefore BE = EB and R(B) = R(E).

Again from Theorem 3, for the EP\_λ-matrix B, we see that B\_+ is a λ-matrix. Since A and B are EP\_λ with R(A) = R(B), we can verify that (AB)\_+ = B\_+ A\_+.

Conversely, if (AB)\_+ is a λ-matrix and AB is EP\_λ then (AB)\_+ is an EP\_λ-matrix. Therefore by Theorem 3, there exists a symmetric idempotent λ-matrix E such that R(AB) = R(E) and (AB) (AB)\_+ = E = (AB)\_+ (AB).

Since \text{rk}(AB) = \text{rk}(A) = r and R(AB) \subseteq R(A), we get R(A) = R(E). Since A is EP\_λ, by Remark 3, it follows that A\_+ is a EP\_λ-matrix. Now by Theorem 3, A\_+ has a λ-matrix \{1\} inverse and \text{rk}(A) = \text{rk}(A^2). Since AB and B are EP\_λ,

\[ R(E) = R(AB) = R((AB)\_+) \subseteq R(B) = R(B) \] implies \text{rk}(AB) = \text{rk}(B) implies R(B) = R(E). Therefore R(A) = R(B). Hence the theorem.

**REMARK 4.** The condition that both A and B are EP\_λ-matrices, is essential in Theorem 5, is illustrated as follows:

Let A = \begin{bmatrix} 1 & \lambda \\ 0 & 0 \end{bmatrix} and B = \begin{bmatrix} 1 & 2\lambda \\ 0 & 0 \end{bmatrix}. A and B are not EP\_1.

A\_+ A = \begin{bmatrix} 1 & \lambda \\ \lambda & 2 \end{bmatrix} has a λ-matrix \{1\} inverse and R(A) = R(B). But AB is not EP\_1. (AB)\_+ = \begin{bmatrix} \frac{1}{1+4\lambda} & 1 \\ \lambda & 2 \end{bmatrix} 0

is not a λ-matrix. Hence the claim.

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