ABSTRACT. In this paper a Γ-group congruence on a regular Γ-semigroup is defined, some equivalent expressions for any Γ-group congruence on a regular Γ-semigroup and those for the least Γ-group congruence in particular are given.

KEY WORDS AND PHRASES. Regular Γ-semigroup, α-idempotent, Right (left) Γ-ideal, Right (left) simple Γ-semigroup, Γ-group, Congruence, Normal family.

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1. INTRODUCTION.

Let S and Γ be two nonempty sets, S is called a Γ-semigroup if for all a, b, c ∈ S, a, b ∈ Γ (i) aab ∈ S and (ii) (aab)bc = a(a(bbc)) hold. S is called regular Γ-semigroup if for any a ∈ S there exist a' ∈ S, a, b ∈ Γ such that a = a'a'ba. We say a' is (a, b)-inverse of a if a = a(a'ba) and a' = a'baa' hold and in this case we write a' ∈ V_α(a). An element e of S is called α-idempotent if eoe = e holds in S. A right (left) Γ-ideal of a Γ-semigroup S is a nonempty subset I of S such that SFI ⊆ I. A Γ-semigroup S is said to be left (right) simple if it has no proper left (right) Γ-ideal. For some fixed a ∈ Γ if we define aob = ab for all a, b ∈ S then S becomes a semigroup. We denote this semigroup by S_a. Throughout our discussion we shall use the notations and results of Sen and Saha [1-2]. For the sake of completeness let us recall the following results of Sen and Saha [1].

THEOREM 1.1. S_a is a group if and only if S is both left simple and right simple Γ-semigroup. (Theorem 2.1 of [1]).

COROLLARY 1.2. Let S be a Γ-semigroup. If S_a is a group for some α ∈ Γ then S_a is a group for all α ∈ Γ. (Corollary 2.2 of [1]).

A Γ-semigroup S is called a Γ-group if S_a is a group for some (hence for all) α ∈ Γ. (Theorem 2.3 of [1]).

THEOREM 1.3. A regular Γ-semigroup S will be a Γ-group if and only if for all α, β ∈ Γ, eaf = fbe = f and eβf = fbe = e for any two idempotents e = eoe and f = fbf of S. (Theorem 3.3 of [1]).

2. Γ-GROUP CONGRUENCES IN A REGULAR Γ-SEMIGROUP.

An equivalence relation ρ on a Γ-semigroup S is called a congruence if (a, b) ∈ ρ implies (c(a, cab)) ∈ ρ and (aα, bαc) ∈ ρ for all a, b, c ∈ S, α ∈ Γ. A congruence ρ in a regular Γ-semigroup S is called Γ-group congruence if S/ρ is a Γ-group (In S/ρ we define a(βα))b(ρ) = a(α báo). Henceforth we shall assume S to be a regular Γ-semigroup and E_α to be its set of α-idempotents.

A family {K_α : α ∈ Γ} of subsets of S is said to be a normal family if
(i) E_α ⊆ K_α for all α ∈ Γ;
(ii) for each a ∈ K_α and b ∈ K_β, aob ∈ K_γ and αb ∈ K_α;
(iii) for each a' ∈ V_α(a) and c ∈ K_γ, αcya' and αγcya' ∈ K_α.
Now let $e \in E_{a}$ and $f \in E_{b}$ and $u \in \Gamma$. Let $x \in V_{0}(ufa)$. Then $faeuf \in E_{u}$. Thus $E_{u} \neq \emptyset$ for all $u \in \Gamma$, consequently $\mathcal{K}_{u} \neq \emptyset$ for all $u \in \Gamma$. We further note that in an orthodox $\Gamma$-semigroup $S$ of Sen and Saha [2] $E_{a} : a \in \Gamma$ is a normal family of $S$.

Let $\mathcal{N}$ be the collection of all normal families $\mathcal{K}_{i}$ of $S(i \in \Lambda)$ where $\mathcal{K}_{i} = \{K_{i} : a \in \Gamma\}$. Let $U_{a} = \bigcap_{i \in \Lambda} K_{i} a$ and $U = \{U_{a} : a \in \Gamma\}$. Then obviously $E_{a} \subseteq U_{a}$. Also if $a \in U_{a}$, $b \in U_{b}$, then $a \in K_{i} a$ for all $i \in \Lambda$, $b \in K_{j} b$ for all $i \in \Lambda$. Thus $aab \in K_{i} a$ and $aab \in U_{a}$ implying $aab \in U_{b}$ and $aab \in U_{a}$. Similarly we can show that if $a' \in V_{0}(a)$ and $c \in U_{a}$ then $a'cay_{i}'$, $acy_{i}'a \in U_{a}$. Thus $U$ is a normal family of subsets of $S$ and $U$ is the least member in $\mathcal{N}$ if we define a partial order in $\mathcal{N}$ by $K_{i} \subseteq K_{j} i f$ $K_{i} \subseteq K_{j}$ for all $a \in \Gamma$. We also observe that when $S$ is orthodox $\Gamma$-semigroup, $U = \{E_{a} : a \in \Gamma\}$.

**THEOREM 2.1.** Let $S$ be a regular $\Gamma$-semigroup. Then for each $K = \{K_{a} : a \in \Gamma\} \in \mathcal{N}$, $\rho_{K} = \{(a,b) \in S \times S : aae = fbb$ for some $a,b \in \Gamma$ and $e \in K_{a}, f \in K_{b}\}$ is a $\Gamma$-group congruence on $S$.

**PROOF.** Let $a \in S$ and $a' \in V_{0}(a)$. Then $aa'(aba) = (aaba)'ba$ implies $a \in \rho_{K}$ for some $a,b \in \Gamma$ such that $aa'(aba) = (aa'(aba))'ba$. Let $b \in V_{0}(b)$ such that $b \in (b\phi fbb)(y(a'\delta a))$. But $b \in V_{0}(b)$, $a' \delta a \in \Gamma$, and so $b \in (b\phi fbb)(y(a'\delta a)) \in \Gamma$, and $b \in \Gamma$. Consequently, $(b,a) \in \rho_{K}$. Now let $(a,b) \in \rho_{K}, (b,c) \in \rho_{K}$. Then there exist $a,b,\gamma,\delta \in \Gamma$, $e \in K_{a}, f \in K_{b}, g \in \Gamma$, such that $ae = fbb$ and $by = hsc$. But $a(a'\gamma b) = (a'a)\gamma b = fbb$. Thus $(a,c) \in \rho_{K}$ and consequently $\rho_{K}$ is an equivalence relation. Let $(a,b) \in \rho_{K}, b \in \Gamma, c \in S$. Then $aa' = fbb$ for some $a,b \in \Gamma$ and some $e \in K_{a}, f \in K_{b}$. Let $c' \in V_{0}(c), y \in V_{0}(b)$, $x \in V_{0}(a)$. Now $a(a'y) = (a'y)ax = (a'y)(a'\delta fbb)(y \delta b) = (a'y)(a'\delta fbb)(y \delta b)c$. But $a(a'y) = a(a'y)c = a(a'y)c = a(a'y)c$. Again $y \delta (b) = x \in \Gamma \subseteq \Gamma$, and consequently $c'(c = (c\delta((c\delta x)\delta fbb)(y \delta b)c) \in \Gamma$. By a similar argument we can show that $e(a'y) = (a'y)fbb(y \delta b)c \in \Gamma$. Thus $(a,c) \in \rho_{K}$. Also it is immediate from the foregoing by duality that $(c\delta(a'y),c) \in \rho_{K}$. Thus $\rho_{K}$ is a congruence on $S$. Also as $S$ is regular, $S/\rho_{K}$ is a regular $\Gamma$-semigroup. Let $e \in E_{a}, f \in E_{b}$. Then $eaf, fae \in K_{b}, ebf, fbe \in K_{a}$. Now $(eaf)f = (eaf)b$ implies that $(eaf,f) \in \rho_{K}$. Thus $(eaf,f) \in \rho_{K}$ and $(f_{a}f)(e_{a}f) = f_{a}f$. Similarly we can show $(e_{a}f)(f_{a}f) = e_{a}f$ and $(f_{a}f)(e_{a}f) = e_{a}f$. So it follows from Theorem 1.3 that $S/\rho_{K}$ is a $\Gamma$-group. Thus $\rho_{K}$ is a $\Gamma$-group congruence on $S$.

For any normal family $K = \{K_{a} : a \in \Gamma\}$ of $S$, the closure $KW$ of $K$ is the family defined by $KW = \{KW_{y} : y \in \Gamma\}$ where $KW_{y} = \{x \in S : eax \in K_{y}$ for some $a \in \Gamma$ and $e \in K_{a}\}$. We call $K$ closed if $KW = K$.

**THEOREM 2.2.** For each $K \in \mathcal{N}, \rho_{K} = \{(a,b) \in S \times S : a' \in V_{0}(b)\}$.

**PROOF.** Let $(a,b) \in \rho_{K}$. Then $fbae$ for some $a,b \in \Gamma$ and $e \in K_{a}, f \in K_{b}$. Then $fbae = fbae_{b} = bae_{b} \in K_{b}$ for some $b' \in V_{0}(b)$. Consequently $a' \in V_{0}(b)$. Conversely, let $a' \in V_{0}(b)$ for some $b' \in V_{0}(b)$. Then $a' \in V_{0}(b)$ for some $a' \in \Gamma$ and $e \in K_{a}$. Therefore $e \in V_{0}(b)$ where $e \in K_{b}$. So $b(a'\delta e) = b(a'\delta e)$, for some $a' \in V_{0}(a)$ where $b(a'\delta e)(a' \delta e) = b(a'\delta e)$. Consequently $(a,b) \in \rho_{K}$.

For any congruence $\rho$ on $S$, let $\ker \rho = \{\ker \rho_{a} : a \in \Gamma\}$ where $(\ker \rho_{a}) = \{x \in S : epx$ for some $e \in E_{a}\}$. 


LEMMA 2.3. For any \( K \in N \), \( \ker \rho_K = KW \).

PROOF. To prove \( \ker \rho_K = KW \), we are to show that \( (\ker \rho_K)_a = (K \cap a^-)_a \) for all \( a \in \Gamma \).

For this let \( x \in (\ker \rho_K)_a \) for some \( a \in \Gamma \). Then \( e \rho_K x \) for some \( e \in E_a \); that is, \( e(x) \in K \gamma \) for some \( \gamma \in \Gamma \), \( e \in E_a \), \( f \in E_b \), \( g \in K \gamma \). So \( g(x) \in K_a \) as \( e \in E_a \). Thus \( x \in (K \cap a^-)_a \). Next let \( x \in (K \cap a^-)_a \). Then \( g(x) \in K \gamma \) for some \( \gamma \in \Gamma \) and \( g \in K \gamma \). Now for some \( e \in E_a \), \( e(x) = (e \rho_K x) \) where \( e \rho_K x \in K_a \) and \( e \rho_K x \in \gamma \gamma \). Thus \( e \rho_K x \). Consequently \( x \in (\ker \rho_K)_a \). So \( (\ker \rho_K)_a = (K \cap a^-)_a \) for all \( a \in \Gamma \).

Let \( K \in N \) and suppose \( a \rho K b \in (K \cap a^-)_a \) for some \( b' \in V^{3}_{Y(b)} \). Then \( a \rho K b \in K_a \). Then for any \( a' \in V^{3}_{B}(a) \), \( a' \rho (a \rho K b) = (a \rho K b) \rho (a' \rho K b) \).

LEMMA 2.4. For each \( K \in N \), \( a \rho K b \) iff one of the following equivalent conditions hold.

(i) \( a \rho (K \cap a^-)_a \) for some \( b' \in V^{3}_{Y(b)} \).

(ii) \( b' a \rho (K \cap a^-)_a \) for some \( b' \in V^{3}_{Y(b)} \).

(iii) \( a' \rho b \) for some \( a' \in V^{3}_{B}(a) \).

(iv) \( b a' \rho (K \cap a^-)_a \) for some \( a' \in V^{3}_{B}(a) \).

Let \( N \) denote the collection of all closed families in \( N \), then \( N \subseteq N \).

THEOREM 2.5. The mapping \( \rho_K \) is a one to one mapping from \( N \) onto the set of \( \Gamma \)-group congruences on \( S \).

PROOF. Let \( \rho \) be a \( \Gamma \)-group congruence on \( S \). Let us denote \( \ker \rho \) by \( K \).

Let \( K \) and \( (\ker \rho)_a \) by \( K_a \). Then \( K_a = \{ x \in S : x e \rho K a e \} \).

Let \( a \in K_a \), \( b \in K_{B} \) then \( a e b \) if \( \rho(a) \cap \rho(b) \neq \emptyset \). Thus \( a e b \) if \( \rho(a) \cap \rho(b) \neq \emptyset \). Thus \( a e b \) if \( \rho(a) \cap \rho(b) \neq \emptyset \).

Next let \( a \in V^{B}_{(a)} \) and \( c \in K_{Y} \). Then \( c \rho K a \) where \( \rho(c) \cap \rho(a) \neq \emptyset \). Thus \( c \rho K a \) where \( \rho(c) \cap \rho(a) \neq \emptyset \).

Therefore \( K \) is a normal family of subsets of \( S \). Next \( (K \cap a^-)_a \subseteq K_{a} \subseteq K_{a} \). Hence \( a \rho K b \) if \( a \rho K b \) if \( a \rho K b \) if \( a \rho K b \) if \( a \rho K b \).

Therefore \( K = KW \) and so \( K = \rho \in \bar{N} \). Thus if \( \rho \) is a \( \Gamma \)-group congruence, then \( \ker \rho = K \subseteq \bar{N} \). We shall now prove that \( \rho_K = \rho \).

Let \( (a, b) \in \rho_K \) and \( (a, b) \in \rho_K \).

THEOREM 2.6. The mapping \( \rho_K \) is a one to one mapping from \( N \) onto the set of \( \Gamma \)-group congruences on \( S \).

PROOF. Let \( \tau \) be a \( \Gamma \)-group congruence on \( S \). By the proof of Theorem 2.5 \( \tau = \rho \), where \( K = \ker \tau \subseteq N \). Thus each \( \Gamma \)-group congruence is of the form \( \rho_K \) for some \( K \in N \subseteq N \).
Thus by lemma 2.3 we have,

**THEOREM 2.6.** The least $\Gamma$-group congruence $\sigma$ on $S$ is given by $\sigma = \alpha \cup$ and ker$\sigma = \mathbb{U}$.

**THEOREM 2.7.** For any $\Gamma$-group congruence $\rho_k$ with $k$ in $\mathbb{N}$, on a regular $\Gamma$-semigroup, the following are equivalent.

(i) $a \rho_k b$.
(ii) $a \alpha \chi \rho \beta b$ for some $x \in K_\mu (\mu \in \Gamma)$ and some (all) $b' \in V^\delta_\gamma (b)$.

(iii) $a' \phi x \rho \ nu b$ for some $x \in K_\mu (\mu \in \Gamma)$ and some (all) $a' \in V^\phi_\gamma (a)$.
(iv) $b' \delta x \rho a$ for some $x \in K_\mu (\mu \in \Gamma)$ and some (all) $b' \in V^\phi_\gamma (b)$.

(vi) $a = \lambda b = f_\beta b$ for some $a, b \in \Gamma$ and some $e \in K_\alpha$, $f \in K_\beta$.
(vii) $a e a = b \beta f$ for some $a, b \in \Gamma$ and some $e \in K_\alpha$, $f \in K_\beta$.

PROOF. (ii) $\Rightarrow$ (iii) Suppose $a \chi \rho b'$ for some $x \in K_\mu$ and $b' \in V^\delta_\gamma (b)$. Then for any $a' \in V^\phi_\gamma (a)$, $a' \alpha \chi \mu \chi b = (a' \phi a) \mu (x \chi (b' \delta b)) \in K_\theta$ as $a' \phi a \in K_\theta$ and $x \chi b' \delta b \in K_\mu$.

(iii) $\Rightarrow$ (vi) Let $a' \phi x \rho b$ for some $x \in K_\mu$ and $b' \in V^\phi_\gamma (b)$. Then $a \alpha b' \rho a$ which is (vi) as $a' \phi x \rho b \in K_\theta$ and $a \alpha b' \rho a = K_\theta$.

(vi) $\Rightarrow$ (vii) Let $a e = \lambda b = f_\beta b$ for some $a, b \in \Gamma$ and some $e \in K_\alpha$, $f \in K_\beta$. Then we have $f \rho b a e = f \rho b a e$ implying $K_\beta \rho b a K_\alpha \rho \beta b \rho a K_\alpha \neq \phi$.

(vii) $\Rightarrow$ (ii) Let $K_\beta \rho b a K_\alpha \rho \beta b K_\rho \beta b K_\alpha \neq \phi$. Then $x \rho b a y = x_1 \rho b a y_1$ for some $x, x_1 \in K_\beta$, $y, y_1 \in K_\alpha$. If $a' \in V^\phi_\gamma (a)$, $b' \in V^\delta_\gamma (b)$, then $a' \phi b \rho a \chi \nu b' \nu \rho y \rho a$ and $a' \phi b \rho a \chi \nu b' \nu \rho y \rho a \chi \nu b' \nu \rho y \rho a$ and we have, $a' \phi b \rho a \chi \nu b' \nu \rho y \rho a = (a' \phi b \rho a \chi \nu b' \nu \rho y \rho a) = (a \phi a) \chi (x \chi (b' \delta b) \nu y \rho a) = (a \phi a) \chi (x \chi (b' \delta b) \nu y \rho a)$ as $b' \chi (x \chi (b' \delta b) \nu y \rho a) \in K_\gamma$.

Thus (ii), (iii), (vi) and (vii) are equivalent.

Interchanging the roles of $a$ and $b$ we see that (iv), (v), (vii) and (viii) are equivalent. Also (i) and (vi) are equivalent by Theorem 2.1. Thus all the conditions (i) (viii) are equivalent.

**COROLLARY 2.8.** Let $\sigma$ denote the least $\Gamma$-group congruence on a regular $\Gamma$-semigroup $S$. Then the following are equivalent.

(i) $a \sigma b$.
(ii) $a \chi \sigma b'$ for some $x \in U_\mu (\mu \in \Gamma)$ and some (all) $b' \in V^\delta_\gamma (b)$.

(iii) $a' \phi x \sigma \nu b$ for some $x \in U_\mu (\mu \in \Gamma)$ and some (all) $a' \in V^\phi_\gamma (a)$.

(iv) $b' \delta x \sigma a$ for some $x \in U_\mu (\mu \in \Gamma)$ and some (all) $b' \in V^\phi_\gamma (b)$.

(vi) $a = \lambda b = f_\beta b$ for some $a, b \in \Gamma$ and some $e \in U_\alpha$, $f \in U_\beta$.
(vii) $a e a = b \beta f$ for some $a, b \in \Gamma$ and some $e \in U_\alpha$, $f \in U_\beta$.

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