ALMOST COMPLEX SURFACES IN THE NEARLY KAHLER S^6

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ABSTRACT: It is shown that a compact almost complex surface in S^6 is either totally geodesic or the minimum of its Gaussian curvature is less than or equal to 1/3.

KEY WORDS AND PHRASES. Almost complex surfaces, nearly Kaehler structure, totally geodesic submanifold, Gaussian curvature.

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1. INTRODUCTION.

The six dimensional sphere S^6 has almost complex structure J which is nearly Kaehler, that is, it satisfies (\nabla_X J)(X) = 0, where \nabla is the Riemannian connection on S^6 corresponding to the usual metric g on S^6. Sekigawa [1] has studied almost complex surfaces in S^6 and has shown that if they have constant curvature K, then either K = 0, 1/6 or 1. Under the assumption that the almost complex surface M in S^6 is compact, he has shown that if K > 1/6, then K = 1 and if 1/6 < K < 1, then K = 1/6. Dillen et al [2-3] have improved this result by showing if 1/6 < K < 1, then either K = 1/6 or K = 1 and if 0 < K < 1/6, then either K = 0 or K = 1/6. However, using system of differential equations (1) (cf. [5], p. 67) one can construct examples of almost complex surfaces in S^6 whose Gaussian curvature takes values outside [9,1/6] or [1/6,1]. The object of the present paper is to prove the following:

THEOREM 1. Let M be a compact almost complex surface in S^6 and K_0 be the minimum of the Gaussian curvature of M. Then either M is totally geodesic or K_0 \leq 1/3.

2. MAIN RESULTS. Let M be a 2-dimensional complex submanifold of S^6 and g be the induced metric on M. The Riemannian connection \nabla of S^6 induces the Riemannian connection \nabla on M and the connection \nabla^perp in the normal bundle \nu. We have the Gauss and Weingarten formulae

\nabla_X Y = \nabla_X Y + h(X, Y), \quad \nabla_X N = -A_N X + \nabla^perp_X N, \quad X, Y \in \mathfrak{G}(M), \quad N \in \nu,

(2.1)

where h, A_N are the second fundamental forms satisfying g(h(X, Y), N) = g(A_N X, Y) and \mathfrak{G}(M) is the Lie-algebra of vector fields on M. The curvature tensors \bar{R}, R and R^perp of the connections \nabla,
\( \nabla \) and \( \nabla^\perp \) respectively satisfy
\[
R(X, Y; Z, W) = R(X, Y; Z, W) + g(h(Y, Z), h(X, W)) - g(h(X, Z), h(Y, W)) \tag{2.2}
\]
\[
\bar{R}(X, Y; N_1, N_2) = R^\perp (X, Y; N_1, N_2) - g([A_{N_1}, A_{N_2}](X), Y) \tag{2.3}
\]
\[
[\bar{R}(X, Y)Z]^\perp = (\bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_Y h)(X, Z), \quad X, Y, Z, W \in \mathfrak{S}(M), \quad N_1, N_2 \in \nu,
\tag{2.4}
\]
where \( [\bar{R}(X, Y)Z]^\perp \) is the normal component of \( \bar{R}(X, Y)Z \), and
\[
(\bar{\nabla}_X h)(Y, Z) = \bar{\nabla}_X h(Y, Z) - h(Y, \nabla_X Z).
\]
The curvature tensor \( \bar{R} \) of \( S^6 \) is given by
\[
\bar{R}(X, Y; Z, W) = g(Y, Z)g(X, W) - g(X, Z)g(Y, W). \tag{2.5}
\]

**Lemma 1.** Let \( M \) be a 2-dimensional complex submanifold of \( S^6 \). Then \( (\bar{\nabla}_X J)(Y) = 0, \quad X, Y \in \mathfrak{S}(M) \).

**Proof.** Take a unit vector field \( X \in \mathfrak{S}(M) \). Then \( \{X, JX\} \) is orthonormal frame on \( M \).
Since \( S^6 \) is nearly Kaehler manifold we have \( (\bar{\nabla}_X J)(X) = 0 \), and \( (\bar{\nabla}_X J)(JX) = 0 \). Also
\[
(\bar{\nabla}_X J)(JX) = -J(\bar{\nabla}_X J)(X) = 0 \quad \text{and} \quad (\bar{\nabla}_X J)(X) = -(\bar{\nabla}_X J)(JX) = 0.
\]
Now for any \( Y, Z \in \mathfrak{S}(M) \), we have \( Y = aX + bJX \) and \( Z = cX + dJX \), where \( a, b, c \) and \( d \) are smooth functions. We have
\[
(\bar{\nabla}_X J)(Z) = a(\bar{\nabla}_X J)(Z) + b(\bar{\nabla}_Y J)(Z) = -a(\bar{\nabla}_Z J)(X) - b(\bar{\nabla}_Z J)(JX)
\]
\[
= -ac(\bar{\nabla}_X J)(X) - ad(\bar{\nabla}_X J)(JX) - bc(\bar{\nabla}_Y J)(JX) - bd(\bar{\nabla}_Y J)(JX) = 0.
\]

**Lemma 2.** For a 2-dimensional complex submanifold \( M \) of \( S^6 \), the following hold
\begin{enumerate}
\item[(i)] \( h(X, JY) = h(JX, Y) = Jh(X, Y) \), \quad \( \bar{\nabla}_X JY = J\bar{\nabla}_X Y \),
\item[(ii)] \( JA_N X = A_{JN} X, \quad A_N JX = -J A_N X \),
\item[(iii)] \( (\bar{\nabla}_X h)(Y, Z) = (\bar{\nabla}_Y h)(JY, Z) = (\bar{\nabla}_Y h)(Y, JZ) \),
\item[(iv)] \( R(X, Y)JZ = J R(X, Y)Z, \quad X, Y, Z \in \mathfrak{S}(M), \quad N \in \nu \).
\end{enumerate}

**Proof.** (i) follows directly from Lemma 1 and equation (2.1). The second part of (ii) follows from (i). For first part of (ii), observe that for \( N \in \nu \) and \( X \in \mathfrak{S}(M) \),
\[
g((\bar{\nabla}_N J)(N), Y) = -g(N, (\bar{\nabla}_N J)(Y)) = 0 \quad \text{for each} \quad Y \in \mathfrak{S}(M),
\]
that is, \( (\bar{\nabla}_N J)(N) \) is normal to \( M \). Hence expanding \( (\bar{\nabla}_N J)(N) \) using (2.1) and equating the tangential parts we get the first part of (ii).

From equations (2.4) and (2.5), we get
\[
(\bar{\nabla}_X h)(Y, Z) = (\bar{\nabla}_Y h)(X, Z) = (\bar{\nabla}_Z h)(X, Y), \quad X, Y, Z \in \mathfrak{S}(M). \tag{2.6}
\]
Also from (i) we have
\[
(\bar{\nabla}_X h)(JY, Z) = (\bar{\nabla}_Y h)(Y, JZ), \quad X, Y \in \mathfrak{S}(M). \tag{2.7}
\]
Thus from (2.6) and (2.7), we get that
\[
(\bar{\nabla}_X h)(JY, Z) = (\bar{\nabla}_X h)(Y, JZ) = (\bar{\nabla}_Y h)(X, JZ) = (\bar{\nabla}_Y h)(JX, Z) = (\bar{\nabla}_X h)(Y, Z),
\]
this together with (2.7) proves (iii). The proof of (iv) follows from second part of (i).
The second covariant derivative of the second fundamental form is defined as

\[ (\nabla^2 h)(X, Y, Z, W) = \nabla_X (\nabla h)(Y, Z, W) - (\nabla h)(\nabla_X Y, Z, W) - (\nabla h)(Y, \nabla_X Z, W) - (\nabla h)(Y, Z, \nabla_X W), \]

where \((\nabla h)(X, Y, Z) = (\nabla_X h)(Y, Z), X, Y, Z, W \in \mathfrak{X}(M)\).

Let \(\Pi: UM \to M\) and \(UM_p\) be the unit tangent bundle of \(M\) and its fiber over \(p \in M\) respectively. Define the function \(f: UM \to R\) by \(f(U) = \| h(U, U) \|^2\).

For \(U \in UM_p\), let \(\sigma_U(t)\) be the geodesic in \(M\) given by the initial conditions \(\sigma_U(0) = p, \dot{\sigma}_U(0) = U\). By parallel translating \(V \in UM_p\) along \(\sigma_U(t)\), we obtain a vector field \(V_U(t)\). We have the following Lemma (cf. [5]).

**Lemma 3.** For the function \(f_U(t) = f(V_U(t))\), we have

1. \(\frac{df_U(t)}{dt} = 2g((\nabla h)(\dot{\sigma}_U, V_U, V_U), h(V_U, V_U))(t)\).
2. \(\frac{d^2 f_U(t)}{dt^2} = 2g((\nabla h)(U, U, V, V), h(V, V)) + 2\| (\nabla h)(U, V, V) \|^2\).

**3. Proof of the Theorem 1.** Since \(UM\) is compact, the function \(f\) attains maximum at some \(V \in UM\). From (i) of Lemma 2, \(\| h(V, V) \|^2 = \| h(JV, JV) \|^2\) and thus we have \(\frac{d^2 f(V)}{dt^2} \leq 0\) and \(\frac{d^2 f_JV}{dt^2} \leq 0\). Using (iii) of Lemma 2 in (2.8) we get that

\[ (\nabla^2 h)(JV, JV, V, V) = (\nabla^2 h)(JV, JV, V, V). \]

The above equation together with the Ricci identity gives

\[ (\nabla^2 h)(JV, JV, V, V) - (\nabla^2 h)(JV, JV, JV, V) = R^{\perp} (JV, V)h(JV, V), \]

Taking inner product with \(h(V, V)\) and using (iv) of Lemma 2, we get

\[ g((\nabla^2 h)(JV, JV, V, V) - (\nabla^2 h)(JV, JV, JV, V), h(V, V)) = R^{\perp} (JV, V)h(JV, V) - 2g(h(R(JV, V), JV, V), h(V, V)). \]

Now using (i) of Lemma 2, we find that \(g(h(U, V), h(U, JV)) = 0\), that is, \(g(A_h(U, U), JV) = 0\) for all \(U \in UM_p\). Since \(\dim M = 2\), it follows that \(A_h(U, U) = \lambda U\). To find \(\lambda\), we take inner product with \(U\) and obtain \(\lambda = \| h(U, U) \|^2\). Thus, \(A_h(U, U) = \| h(U, U) \|^2 U\). From equations (2.2) and (2.5) we obtain

\[ R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + A_h(Y, Z)X - A_h(X, Z)Y, \]

which gives

\[ R(JV, V)JV = -V + A_h(V, JV) - A_h(JV, V)V = -V + 2A_h(V, V) = -V + 2\| h(V, V) \|^2 V. \]

Also from (2.3) and (2.5) we get

\[ R^{\perp} (JV, V, h(JV, V), h(V, V)) = g([A_h(JV, V), A_h(V, V)], JV, V) = -2g(A_h(V, V), A_h(V, V)) = -2\| h(V, V) \|^4. \]
Substituting (3.2) and (3.3) in (3.1) we get

\[ g((\nabla^2 h)(JV, JV, V, V) - (\nabla^2 h)(V, JV, JV, V), h(V, V)) = 2f(V)(1 - 3f(V)). \]  

(3.4)

From (iii) of Lemma 2, it follows that

\[ (\nabla h)(JV, JV, V) = (\nabla h)(J^2 V, V, V) = -(\nabla h)(V, V, V), \]

this together with \( \nabla XY = J \nabla Y \) of (i) in Lemma 2, gives

\[ (\nabla^2 h)(V, JV, JV, V) = -(\nabla^2 h)(V, V, V). \]

Using this and (ii) of Lemma 3 in (3.4), we obtain

\[ \frac{d^2}{dt^2} f_V(0) + \frac{d^2}{dt^2} f_{JV}(0) = 2f(V)(1 - 3f(V)) + 2 \| (\nabla h)(V, V, V) \|^2 + 2 \| (\nabla h)(JV, V, V) \|^2 \leq 0 \]

Thus either \( f(V) = 0 \), that is, \( M \) is totally geodesic or \( 1/3 \leq f(V) \). Since an orthonormal frame of \( M \) is of the form \( (U, JU) \), the Gaussian curvature \( K \) of \( M \) is given by

\[ K = 1 + g(h(U, U), h(JU, JU)) - g(h(JU, JU), h(U, U)) = 1 - 2 \| h(U, U) \|^2. \]

Thus \( K:UM \to R \), is a smooth function, and \( UM \) being compact, \( K \) attains its minimum \( K_0 = \min K \) and we have \( K_0 = 1 - 2 \max \| h(U, U) \|^2 \), from which for the case \( 1/3 \leq f(V) \), we get \( K_0 \leq 1/3 \). This completes the proof of the Theorem.

As a direct consequence of our Theorem we have

**COROLLARY.** Let \( M \) be a compact almost complex surface in \( S^6 \). If the Gaussian curvature \( K \) of \( M \) satisfies \( K > 1/3 \), then \( M \) is totally geodesic.

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