ABSTRACT. A connection between the numerical range of the multiparameter linear system \( P(\lambda) \) and the joint numerical range of the (commuting) separating operator system is given.

KEY WORDS AND PHRASES. Multiparameter operator system, uniform reasonable cross-norm, numerical range, and spectral problems.

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1. INTRODUCTION. We consider the following multiparameter linear operator system:

\[
P(\lambda) = (P_1(\lambda), \ldots, P_n(\lambda)),
\]

where

\[
P_1(\lambda) = B_1 - \lambda_1 A_{11} - \cdots - \lambda_n A_{1n},
\]
\[
P_2(\lambda) = B_2 - \lambda_1 A_{21} - \cdots - \lambda_n A_{2n},
\]
\[
\ldots \ldots \ldots \ldots \ldots \ldots \ldots \,
\]
\[
P_n(\lambda) = B_n - \lambda_1 A_{n1} - \cdots - \lambda_n A_{nn},
\]

and the operators \( B_j \) and \( A_{jk} \) acting on different (smooth) complex Banach spaces \( X_j \) are assumed to be bounded and linear \((j, k = 1, \ldots, n)\).

DEFINITION 1.1. Let \( X \) be a (smooth) complex Banach space. Then in this smooth space there is a unique map \( \phi: X_1 \to X_1 \) such that (see [1]).

\[
\| \phi(x_1) \| = \| x_1 \|, \quad < x_1, \phi(x_1) > = \| x_1 \|^2, \quad \text{for} \ x_1 \in X_1.
\]

Now using the function \( \phi \), we can define a semi-inner product on \( X_1 \) by

\[
[x_1, y_1]_1 = < x_1, \phi(y_1) >_1, \quad x_1, y_1 \in X_1.
\]

DEFINITION 1.2. Let \( \hat{X} = X_1 \otimes_\alpha \cdots \otimes_\alpha X_n \) denote the completion of the tensor product \( X_1 \otimes \cdots \otimes X_n \) with respect to a uniform reasonable cross-norm \( \alpha \). For details, see [2]. Let \([.,.]_j\) be a semi-inner product on a Banach space \( X_j \) with respect to a norm \( \alpha_j \). Then there is a semi-inner product \([.,.] \) on \( \hat{X} \), defined by

\[
[x,y] = [x_1,y_1]_1 \cdots [x_n,y_n]_n,
\]

where \( x = x_1 \otimes \cdots \otimes x_n \in \hat{X} \), and \( y = y_1 \otimes \cdots \otimes y_n \in \hat{X} \).

Let \( P_j(\lambda_1, \ldots, \lambda_n) \) be bounded linear operators on \( X_j, 1 \leq j \leq n \).
DEFINITION 1.3. We define the (spatial) numerical range of the system $P(\lambda)$ as the set

$$V[P(\lambda_1),...,P(\lambda_n)] = \bigcup_{j=1}^{n} \{ (\lambda_1,...,\lambda_n) \in \mathbb{C}^n : [P_j(\lambda) x_j, x_j]_{j=1} = 0 \},$$

where $[x_j, x_j]_{j=1} = 1$.

We next introduce the operators which arise in the separation of the spectral parameters.

To each operator $T_j : X_j \rightarrow X_j$, we associate an operator acting on $\hat{X}$ given by

$$T_j^\otimes = I_1 \otimes ... \otimes I_{j-1} \otimes T_j \otimes I_{j+1} \otimes \cdots \otimes I_n.$$

In order to study the linear operator system $P(\lambda)$, we construct the operators $\Delta_0, \Delta_1,..., \Delta_n$, which are well-defined determinants of the operator matrix $(A_{ij})_{i,j=1}^{n}$ and the matrices obtained from this matrix by replacing the $j$-th column by the column of operators $B_1,...,B_n$, $j = 1,...,n$. For details, see [3,4].

As in the multiparameter spectral theory, the separation of the spectrum is understood to mean the reduction of the spectral problems for the system $P(\lambda)$ to the spectral problems for the commuting operator system $(\Delta_0^{-1} \Delta_1 - \lambda_1 I,...,\Delta_0^{-1} \Delta_n - \lambda_n I)$, the system $(\Delta_0^{-1} \Delta_j - \lambda_j I)$, $1 \leq j \leq n$, is called a separating system.

In the present problem, we consider the case when the operator $\Delta_0$ is invertible, and the separating system $(\Delta_0^{-1} \Delta_j - \lambda_j I)$ is commutative ($1 \leq j \leq n$). The aim of this note is to announce the following result on the multiparameter spectral theory.

2. MAIN RESULTS. Let $(\Delta_0^{-1} \Delta_1 - \lambda_1 I,...,\Delta_0^{-1} \Delta_n - \lambda_n I)$ be a commutative family of operators acting on $\hat{X}$.

DEFINITION 2.1. We introduce the concept of the (spatial) numerical range of the operator system $(\Delta_0^{-1} \Delta_j - \lambda_j I)$, $1 \leq j \leq n$, as the set

$$v_\otimes [\Delta_0^{-1} \Delta_1 - \lambda_1 I,...,\Delta_0^{-1} \Delta_n - \lambda_n I] = \{ \lambda = (\lambda_1,...,\lambda_n) \in \mathbb{C}^n : [\Delta_0^{-1} \Delta_j x, x] = \lambda_j [x, x] \},$$

for $x = x_1 \otimes ... \otimes x_n \in \hat{X}$, and $[x, x] = 1$.

We now write the result connecting the numerical range of the system $P(\lambda)$ and the joint numerical range of the (commuting) separating operator system $(\Delta_0^{-1} \Delta_j - \lambda_j I)$, $1 \leq j \leq n$.

THEOREM 2.1. $V[P(\lambda_1),...,P(\lambda_n)] = v_\otimes [\Delta_0^{-1} \Delta_1 - \lambda_1 I,...,\Delta_0^{-1} \Delta_n - \lambda_n I]$.

REFERENCES