ON THE THREE-DIMENSIONAL CR-SUBMANIFOLDS OF THE SIX-DIMENSIONAL SPHERE

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ABSTRACT. We show that the six-dimensional sphere does not admit three-dimensional totally umbilical proper CR-submanifolds.

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1. INTRODUCTION. The six-dimensional unit sphere $S^6(1)$ has a nearly Kaehler structure $J$ constructed in a natural way by making use of Cayley division algebra [3]. It is because of this nearly Kaehler, non-Kaehler structure, that $S^6(1)$ has drawn the attention. In particular, almost complex submanifolds, CR-submanifolds and totally real submanifolds of $S^6(1)$ have been considered by A. Gray [4], K. Sekigawa and N. Ejiri [2]. For three-dimensional totally real submanifolds of $S^6(1)$ of constant curvature, N. Ejiri proved the following [2].

THEOREM 1. Let $M$ be a 3-dimensional totally real submanifold of constant curvature $c$ in $S^6(1)$. Then $c = 1$ (totally geodesic) or $c = \frac{1}{15}$ (minimal).

In this paper we consider 3-dimensional CR-submanifolds of $S^6(1)$. We prove the following result:

THEOREM 2. There are no 3-dimensional totally umbilical proper CR-submanifolds in $S^6(1)$.

2. PRELIMINARIES.

Let $C_+$ be the set of all purely imaginary Cayley numbers. The $C_+$ can be viewed as a 7-dimensional linear subspace $\mathbb{R}^7$ of $\mathbb{R}^8$. Consider the unit hypersphere which is centered at the origin

$$S^6(1) = \{x \in C_+ | < x, x > = 1\}.$$

The tangent space $T_x S^6$ of $S^6(1)$ at a point $x$ may be identified with the affine subspace of $C_+$ which is orthogonal to $x$. On $S^6(1)$ define a (1,1)-tensor field $J$ by putting

$$J_x U = x \times U,$$

where the above product is defined as in [3] for $x \in S^6(1)$ and $U \in T_x S^6$. 
The above tensor field $J$ determines an almost complex structure (i.e., $J^2 = -Id$) on $S^6(1)$. The compact simple lie group of automorphisms $G_2$ acts transitively on $S^6(1)$ and preserves both $J$ and the standard metric on $S^6(1)$, [3].

Now let $G$ be the $(2,1)$-tensor field on $S^6(1)$ defined by

$$G(X,Y) = (\nabla_X J)Y$$

where $\nabla$ is the Levi-Civita connection on $S^6(1)$ and $X, Y \in T_x S^6$.

Since $\nabla_X J$ is skew-symmetric with respect to the Hermitian metric $g$ on $S^6(1)$, it follows that $G$ has the following property

$$g(G(X,Y), Z) + g(G(X,Z), Y) = 0 \quad (2.1)$$

where $X, Y, Z \in \mathfrak{X}(S^6)$.

A submanifold $M$ of of $\dim (2p + q)$ in $S^6(1)$ is called a CR-submanifold if there exists a pair of orthogonal complementary distributions $D$ and $\overline{D}$ such that $JD = D$ and $J\overline{D} \subset \nu$, where $\nu$ is the normal bundle of $M$ and $\dim D = 2p$, $\dim \overline{D} = q$[1]. Thus the normal bundle $\nu$ splits as $\nu = JD \oplus \mu$, where $\mu$ is invariant sub-bundle of $\nu$ under $J$.

A CR-submanifold is said to be proper if neither $D = \{0\}$ nor $\overline{D} = \{0\}$.

We denote by $\nabla, \nabla, \nabla$ the Riemannian connections on $M$, $S^6$ and the normal bundle, respectively. They are related by Gauss formula and Weingarten formula:

$$\nabla_X Y = \nabla_X Y + h(X,Y) \quad (2.2)$$

$$\nabla_X N = -A_N X + \nabla_X N \quad N \in \nu \quad (2.3)$$

where $h(X, Y)$ and $A_N X$ are the second fundamental forms which are related by

$$g(h(X, Y), N) = g(A_N X, Y) \quad (2.4)$$

$X$ and $Y$ are vector fields on $M$.

Now a CR-submanifold is said to be totally umbilical if $h(X, Y) = g(X, Y)H$ where $H = \frac{1}{\mu} (\text{trace } h)$ is the mean curvature vector. If $M$ is a totally umbilical CR-submanifold, then equations (2) and (3) become

$$\nabla_X Y = \nabla_X Y + g(X,Y)H \quad (2.5)$$

$$\nabla_X N = -g(H, N)X + \nabla_X N \quad (2.6)$$

Let $R$ be the curvature tensor associated with $\nabla$. Then the equation of Gauss is given by

$$R(X, Y; Z, W) = g(X, Z)g(Y, W) - g(Y, Z)g(X, W) + g(h(X, Z), h(Y, W)) - g(h(Y, Z), h(X, W))$$

It is known that for $X, Y$ in $D$, $G(X,Y) = 0$, and $G(W, W) = 0$ for all $W \in \mathfrak{X}(S^6)$.

3. 3-DIMENSIONAL CR-SUBMANIFOLDS OF $S^6(1)$:

Let $M$ be a 3-dimensional totally umbilical proper CR-submanifold of $S^6(1)$. Since $M$ is proper, $D \neq \{0\}$ and $\overline{D} \neq \{0\}$. Then since $\dim M = 3$, we have $\dim D = 2$ and $\dim \overline{D} = 1$.

We have the following:

**Lemma 1.** If $M$ is a 3-dimensional totally umbilical proper CR-submanifold of $S^6(1)$, then $H \in JD$.

**Proof.** For $X, Y \neq 0$ in $D$ we use equation (2.5) and the equation $J \nabla_X Y = \nabla_X JY$ to get
Taking inner product in (3.1) with $N \in \mu$ we have

$$g(X, Y) g(JH, N) = g(X, JY) g(H, N)$$

(3.2)

In particular, if we let $Y = JX$ in (3.2) we get

$$\|X\| g(H, N) = 0$$

From which it follows that $H \in \mathbf{J} \mathbf{D}$. 

**Lemma 2.** If $M$ is a 3-dimensional totally umbilical CR-submanifold of $S^6(1)$, the $\|H\|$ is constant.

**Proof.** Using (2.7) and the equation $h(X, Y) = g(X, Y)H$ we get

$$R(X, Y; Z, W) = (1 + \|H\|^2) \{g(X, Z)g(Y, W) - g(Y, Z)g(X, W)\}$$

(3.3)

Then since dim $M = 3$, we invoke Schur’s theorem to conclude that $(1 + \|H\|^2)$ is constant. Thus $\|H\|$ is constant.

**4. Proof of Theorem 2.**

In this section let $\{X, JX, Z\}$ denote an orthonormal frame field for the 3-dimensional totally umbilical CR-submanifold $M$ of $S^6(1)$. The unit vector fields $X, JX$ are in $D$ and the unit vector field $Z$ is in $\mathbf{J} \mathbf{D}$. Since $M$ is totally umbilical, the equation $h(X, Y) = g(X, Y)H$ implies that

$$h(X, JX) = h(X, Z) = h(JX, Z) = 0$$

and

$$h(X, X) = h(JX, JX) = h(Z, Z) = H$$

(4.1)

We know from the previous Lemma that $H \in \mathbf{J} \mathbf{D}$. Since dim $\mathbf{J} \mathbf{D} = 1$, then one can write $H = \alpha JZ$ for some smooth function $\alpha$ on $M$. Therefore

$$h(X, X) = h(JX, JX) = h(Z, Z) = \alpha JZ$$

Using equation (2.4) with $N = JZ$ we get

$$A_{JZ}X = \alpha X, \quad A_{JZ}JX = \alpha JX, \quad A_{JZ}Z = \alpha Z$$

(4.2)

So the frame field $\{X, JX, Z\}$ diagonalizes $A$. Now in $S^6(1)$ we have equation (2.1) i.e.

$$g((\nabla_X J)Y, Z) + g((\nabla_X Z)Y, J) = 0$$

for any $X, Y, Z \in \mathbf{S}^6$. Since for $X, Y \in D (\nabla_X J)Y = 0$, then using this equation with $Y = JX$ for our orthonormal frame field $\{X, JX, Z\}$ in $M$, we get

$$g((\nabla_X J)Z, JX) = 0$$

(4.3)

Using equation (2.5), (4.3) and (2.6) with the fact that $H \in \mathbf{J} \mathbf{D}$ and $(\nabla_X J)Z = \nabla_X JZ - J \nabla_X Z$ we get

$$g(\nabla_X Z, X) = 0$$

(4.4)

Again using equation (2.5) and (2.6) in equation (2.1) with $Y = X$, we get

$$g(\nabla_X Z, JX) = \alpha$$

(4.5)
Also using equation (2.1) and \( (\nabla JX)Z = \nabla JXZ - J \nabla JXZ \) we get

\[
g(\nabla JXZ, X) = -\alpha \tag{4.6}
\]

Switching the role of \( X \) and \( Y \) in equation (2.1) and letting \( Y = JX \) we obtain

\[
g(\nabla JXZ, JX) = 0 \tag{4.7}
\]

Now using the equation \( g((\nabla X)X, JX) = 0 \) and \( g(\nabla JXJX, z) = 0 \) we get

\[
g(\nabla X, Z) = 0, \quad g(\nabla JXJX, Z) = 0 \tag{4.8}
\]

From the equation \( (\nabla Z)Z = 0 \), using equation (4.1) and (4.2) and the fact that \( \nabla Z \in D \), we get

\[
\nabla Z = 0, \quad \nabla JZ = 0 \tag{4.9}
\]

Using equations (4.4), (4.5), (4.6), (4.7), (4.8) and the first part of equation (4.9) we can write the local equations for the frame field \( \{X, JX, Z\} \) as follows:

\[
\nabla X = aJX, \quad \nabla JX = -aX, \quad \nabla Z = 0
\]

\[
\nabla X = aJX, \quad \nabla JX = -bJX + aZ, \quad \nabla Z = cJX
\]

\[
\nabla X = -aX - aZ, \quad \nabla JXJX = bX, \quad \nabla Z = -cX \tag{4.10}
\]

for some smooth functions \( a, b \) and \( c \).

The curvature tensor \( R \) is given by

\[
R(X, Y; Z, W) = < \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla Z, W > \quad [X, Y]
\]

Then using this equation with the help of equations (4.10) we get \( R(X, Z, Z, X) = \alpha^2, \quad \alpha = \|H\| \).

But from equation (3.3) we know that \( R(X, Z, Z, X) = - (1 + \alpha^2) \). This is a contradiction and hence \( S^6(1) \) cannot admit a 3-dimensional totally umbilical proper CR-submanifolds.

REFERENCES