ON RADIi OF CONVEXITY AND STARLIKENESS OF SOME CLASSES OF ANALYTIC FUNCTIONS

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ABSTRACT. Let $P[A,B]$, $-1 \leq B < A \leq 1$, be the class of functions $p$ such that $p(z)$ is subordinate to $\frac{1-Az}{1-Bz}$. Let $P(\alpha)$ be the class of functions with positive real part greater than $\alpha$, $0 < \alpha < 1$. It is clear that $P[A,B] \subseteq P(1) \subseteq P[1,-1]$. The principal results in this paper are the determination of the radius of $\beta$-starlikeness and $\beta$-convexity of $f(z)$ with $\beta = \frac{1-A}{1-B}$, when $f(z)$ is restricted to certain classes of univalent and analytic functions related with $P[A,B]$.

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1. INTRODUCTION.

Let $f$ be analytic in $E = \{z : |z| < 1\}$, and be given by

$$f(z) = z + \sum_{n=2}^\infty a_nz^n.$$  \hspace{1cm} (1.1)

A function $g$, analytic in $E$, is called subordinate to a function $G$ if there exists a Schwarz function $w(z)$, $w(z)$ analytic in $E$ with $w(0) = 0$ and $|w(z)| < 1$ in $E$, such that $g(z) = G(w(z))$.

In [1], Janowski introduced the class $P[A,B]$. For $A$ and $B$, $-1 \leq B < A \leq 1$, a function $p$, analytic in $E$ with $p(0) = 1$ belongs to the class $P[A,B]$ if $p(z)$ is subordinate to $\frac{1-Az}{1-Bz}$.

Also $C[A,B]$ and $S^*[A,B]$ denote the classes of functions, analytic in $E$ and given by (1.1) such that $\frac{d^2}{dz^2} \in P[A,B]$ and $\frac{d^2}{dz^2} \in P[A,B]$ respectively. For $A = 1$, and $B = -1$, we note that $C[1,-1] = C$ and $S^*[1,-1] = S^*$, the classes of convex and starlike functions in $E$. Also $S^*[A,B] \subset S^*[\frac{1-A}{1-B}] \subset S^*[1,-1]$ and $C[A,B] \subset C[\frac{1-A}{1-B}] \subset C[1,-1]$, where $S^*[\frac{1-A}{1-B}]$ and $C[\frac{1-A}{1-B}]$ denote the classes of starlike and convex functions of order $\frac{1-A}{1-B}$ respectively. These classes were first introduced by Robertson in [2].

A function $f$, analytic in $E$ and given by (1.1), is said to be in the class $R[A,B]$, $-1 \leq B < A \leq 1$, if and only if
Hence

\[
\frac{(zf'(z))'}{f'(z)} - \frac{1-A}{1-B} = p(z) + \frac{zp'(z)}{p(z)} - \frac{1-A}{1-B}
\]

Using Lemma 2.3 for \(a = 1 - \beta\), we have for \(R_1 \leq R_2\)

\[
Re \left[ \frac{(zf'(z))'}{f'(z)} - \frac{1-A}{1-B} \right] \geq \frac{1-(3A-B)r+A^2r^2}{(1-A)r(1-Br)} - \frac{1-A}{1-B} \]

\[
= \frac{A-B}{1-B} \left[ \frac{1-(2+A-B)r+Ar^2}{(1-A)r(1-Br)} \right],
\]

and this implies that \(Re \left[ \frac{(zf'(z))'}{f'(z)} - \frac{1-A}{1-B} \right] \geq 0\) for \(|z| < r_0\), where \(r_0\) is given by (3.1). The inequality \(R_1 < R_2\) is satisfied whenever \(T(r) = 1-(2+A-B)r+Ar^2 \geq 0\). But \(T(0) = 1 > 0\) and \(T(1) = B - 1 < 0\). So \(T(r)\) has at least one root in (0,1). Let \(r_0\) given by (3.1) be that root of \(T(r) = 0\). Then in \([0, r_0), R_1 < R_2\) and hence \(f \in C(\frac{1-A}{1-B})\) for all \(z\) with \(|z| = r \leq r_0 < 1\).

This result is sharp for the function \(f_0 \in S^*[A, B]\) such that

\[
\frac{zf_0'(z)}{f_0(z)} = \frac{1+Az}{1+Bz}
\]

**THEOREM 3.2.** Let \(g \in S^*[A, B]\) and let \(\frac{g(z)}{z} \in P[A, B]\). Then \(f \in C(\frac{1-A}{1-B})\) for \(|z| < r_0\), where \(r_0\) is given by (3.1).

**PROOF.** \((zf'(z)) = g(z)p(z), p \in P[A, B]\). This gives us

\[
\frac{(zf'(z))'}{f'(z)} = \frac{g'(z)}{g(z)} + \frac{zp'(z)}{p(z)}
\]

Applying the usual inequalities, we obtain

\[
Re \left[ \frac{(zf'(z))'}{f'(z)} - \frac{1-A}{1-B} \right] \geq \frac{1-Ar}{1-Br} \frac{(A-B)r}{(1-A)r(1-Br)} - \frac{1-A}{1-B} \]

\[
= \frac{(A-B)[1-(2+A-B)r+Ar]}{(1-B)(1-Ar)(1-Br)}
\]

Hence we obtain the required result that \(f \in C(\frac{1-A}{1-B})\) for \(|z| < r_0\) and \(r_0\) is given by (3.1).

**THEOREM 3.3.** Let \(g \in S^*[A, B]\) and \(\frac{g(z)}{z} \in P[A, B]\). Then \(\frac{g(z)}{z} \in P(\frac{1-A}{1-B})\) for \(|z| < r_0\), where \(r_0\) is given by (3.1).

**PROOF.** We have \((zf'(z)) = g(z)p(z), p \in P[A, B]\) and so

\[
\frac{(zf'(z))'}{g'(z)} = p(z) + \frac{g(z)}{zg'(z)} \frac{zp'(z)}{p(z)}
\]

Thus

\[
Re \left[ \frac{(zf'(z))'}{g'(z)} - \frac{1-A}{1-B} \right] \geq Re \left[ p(z) \left( 1 - \frac{(1-Br)}{(1-Ar)(1-Br)} \frac{(A-B)r}{(1-Ar)(1-Br)} \right) - \frac{1-A}{1-B} \right]
\]

\[
= \frac{(1-Ar)}{(1-Br)} \left( 1 - \frac{(3A-B)r+A^2r^2}{(1-Ar)(1-Br)} \right) - \frac{1-A}{1-B} \]

\[
= \frac{(A-B)}{(1-B)} \left[ \frac{1-(2+A-B)r+Ar^2}{(1-Ar)(1-Br)} \right]
\]
Clearly \( k \geq 2 \) and \( R_k[A,B] = S^*[A,B] \). Also \( R_{[1,-1]} = U_k \), the class of functions with bounded radius rotation discussed in [3].

Similarly we can define the class \( V_k[A,B] \) as follows. A function \( f \), analytic in \( E \) and given by (1.1) belongs to \( V_k[A,B] \), \( k \geq 2 \), if and only if

\[
f(z) = \frac{(S_2(z)/z)^{k^{1/2}}}{(S_1(z)/z)^{(k-1)/2}}, \quad S_1, S_2 \in S^*[A,B].
\]

From (1.2) and (1.3), it is clear that

\[
f(z) \in V_k[A,B] \quad \text{if and only if} \quad zf'(z) \in R_k[A,B]
\]

It may be noted that \( V_2[A,B] = C[A,B] \) and \( V_{[1,-1]} = V_1 \), the class of functions of bounded rotation first discussed by Paatero [4].

2. PRELIMINARY RESULTS

**Lemma 2.1** [5] Let \( p \in P[A,B] \). Then

\[
\frac{1-Ar}{1-Br} \leq \Re p(z) \leq \left| p(z) \right| \leq \frac{1+Ar}{1+Br}
\]

The following is the extension of Libera's result [6].

**Lemma 2.2.** Let \( N \) and \( D \) be analytic in \( E \), \( D \) map onto a many-sheeted starlike region. \( N(0) = 0 = D(0) \) and \( \frac{N(z)}{D(z)} \in P[A,B] \). Then \( \frac{N(z)}{D(z)} \in P[A,B] \). For the proof of this result we refer to [5].

**Lemma 2.3.** [7] Let \( p \in P[A,B] \). Then, for \( z \in E \), \( \alpha \geq 0 \) and \( \beta \geq 0 \), we have

\[
\Re \left\{ \alpha p(z) + \beta^2 p'(z) \right\} = \left\{ \begin{array}{ll}
\frac{\alpha - \{\beta(A-B) + 2\alpha A\} r + \alpha^2 r^2}{(1-Ar)(1-Br)}, & R_1 \leq R_2 \\
\frac{\beta A + B}{A-B} + \frac{2[(L_1 K_1)^{1/2} - \beta(1-AB^2)]}{(A-B)(1-r^2)}, & R_2 \leq R_1 
\end{array} \right.
\]

where

\[
R_1 = \left( \frac{L_1}{K_1} \right)^{1/2}, \quad R_2 = \frac{1-Ar}{1-Br}, \quad L_1 = \beta(1-A)(1+Ar^2)
\]

and

\[
K_1 = \alpha(A-B)(1-r^2) + \beta(1-B)(1+Br^2).
\]

This result is sharp.

3. MAIN RESULTS.

**Theorem 3.1.** Let \( f \in S^*[A,B] \). Then \( f \in C \left( \frac{1-A}{1-B} \right) \) for

\[
|z| < r_0 = \frac{2}{(2+A-B)+\sqrt{(2+A-B)^2-4A}}
\]

This result is sharp.

**Proof.** We have \( zf'(z) = f(z)p(z), p \in P[A,B] \)
Hence \( \frac{(f(z))'}{f(z)} \in P\left(\frac{1-A}{1-B}\right) \) for \( |z| < r_0 \), where \( r_0 \) is given by (3.1).

Our next result is about the radius of convexity problem for the class \( V_k[A, B] \).

**THEOREM 3.4.** Let \( f \in V_k[A, B] \), \( k \geq 2 \). Then \( f \in C\left(\frac{1-A}{1-B}\right) \) for \( |z| < r_1 \), where

\[
 r_1 = \frac{4}{k(1-B) + \sqrt{k(1-B)^2 + 16B}} \tag{3.2}
\]

**PROOF.** Since \( f \in V_k[A, B] \), we have from (1.3)

\[
 f'(z) = \frac{(S_1(z)/z)^{1/2}}{(S_2(z)/z)^{1/2}}, \quad S_1, S_2 \in S^*[A, B]
\]

This implies that

\[
 \frac{(zf'(z))'}{f'(z)} = \left(\frac{k + 1}{4 + 2}\right)p_1(z) - \left(\frac{k + 1}{4 - 2}\right)p_2(z), \quad p_1, p_2 \in P[A, B]
\]

so

\[
 \text{Re} \left[ \frac{(zf'(z))'}{f'(z)} \right] \cdot \frac{1-A}{1-B} = \frac{k + 1}{4 + 2} - \frac{k + 1}{4 - 2} - \frac{1-A}{1-B}
\]

Hence \( f \in C\left(\frac{1-A}{1-B}\right) \) for \( |z| < r_1 \), \( r_1 \) is given by (3.2).

From Theorem 3.4 and relation (1.4) we have the following:

**THEOREM 3.5.** Let \( f \in R_\alpha[A, B] \). Then \( f \in S^{*}\left(\frac{1-A}{1-B}\right) \) for \( |z| < r_1 \) where \( r_1 \) is given by (3.2).

**THEOREM 3.6.** Let \( \alpha \) and \( m \) be any positive integers and \( f \in R_\alpha[A, B] \). Then the function \( F \) defined by

\[
 (F(z))^\alpha = \frac{\alpha + m}{z^m} \int_0^z t^{\alpha-1}(f(t))^\alpha dt \tag{3.3}
\]

belongs to \( S^{*}\left(\frac{1-A}{1-B}\right) \) for \( |z| < r_1 \), \( r_1 \) is given by (3.2).

**PROOF.** Let \( J(z) = \int_0^z t^{\alpha-1}(F(t))^\alpha dt \) and so

\[
 (F(z))^\alpha = \frac{\alpha + m}{z^m} J(z),
\]

and

\[
 \alpha zF'(z) \quad \frac{zJ'(z)}{F(z)} = \frac{zJ'(z) - mJ(z)}{J(z)} = \frac{N(z)}{D(z)}
\]

or

\[
 \frac{zF'(z)}{F(z)} = \frac{1}{\alpha} \frac{zJ'(z) - mJ(z)}{J(z)} = \frac{N(z)}{D(z)}
\]

\[
 N(0) = 0 = D(0)
\]
By a result of Bernardi [8] and Theorem 3.5, $D(z)$ is a $(m + \alpha - 1)$-valent starlike function for $|z| < r_1$. Also

$$\frac{N'(z)}{D'(z)} = \frac{1}{\alpha} \left[ \frac{(zJ'(z))'-mJ'(z)}{J'(z)} \right]$$

Now, by Theorem 3.5, $f \in S^\ast\left(\frac{1-A}{1-B}\right)$ for $|z| < r_1$ and this implies that $\frac{N(z)}{D(z)} \in P\left(\frac{1-A}{1-B}\right)$ for $|z| < r_1$. Hence

$$\frac{N(z)}{D(z)} \in P\left(\frac{1-A}{1-B}\right) \quad \text{for} \quad |z| < r_1, \quad \text{see [8]}.$$

This proves our result.

Similarly, we can prove the following:

**THEOREM 3.7.** Let $\alpha$ and $m$ be positive integers and $f \in V_\delta[A,B]$. Let $F$ be defined by (3.3). Then

$f \in C\left(\frac{1-A}{1-B}\right)$ for $|z| < r_1$ where $r_1$ is given by (3.2).

We now prove:

**THEOREM 3.8.** Let $f$ and $g \in R_\delta[A,B]$ and, for $m$ positive integers, let $F$ be defined as

$$F(z) = \frac{(m + \alpha)}{(g(z))^m} \int_0^z t^{(m-1)}(f(t))^\alpha dt \quad (3.4)$$

Then $F \in S^\ast\left(\frac{1-A}{1-B}\right)$ for $|z| < r_0$

where $r_0 = \min(r_1, r_2)$, $r_1$ is given by (3.2) and $r_2$ is the least positive root of the equation

$$\{(1-B)-\alpha(1-A)-\{(A-B)(1+2m)r+\{(A-B)\}r^2 = 0, \quad (3.5)$$

**PROOF.** Let $J_1(z) = \frac{\alpha}{\alpha + m} \int_0^z t^{(m-1)}(f(t))^\alpha dt$.

Then $(F(z))^\alpha = \left(\frac{z}{g(z)}\right)\alpha J_1(z)$, where by Theorem 3.6, $J_1 \in S^\ast\left(\frac{1-A}{1-B}\right)$ for $|z| < r_1$.

So

$$\frac{\alpha zF'(z)}{F(z)} = \frac{zJ_1'(z)}{J_1(z)} + m \left(1 - \frac{zg'(z)}{g(z)}\right)$$

Thus

$$\Re \left[ \frac{zF'(z)}{F(z)} - \frac{1-A}{1-B} \right] \geq \frac{1}{\alpha} \left[ \left(1 + \frac{B-A}{1-B}\right)r/(1+r) \right] + \left[\frac{2m}{\alpha}(B-A)r/(1-r) - \frac{1-A}{1-B}\right]$$

This implies $\Re \left[ \frac{zF'(z)}{F(z)} \right] \geq \frac{1-A}{1-B}$ for $|z| < r_2$, where $r_2$ is the least positive root of (3.5). Hence $F \in S^\ast\left(\frac{1-A}{1-B}\right)$ for $|z| < r_0$, where $r_0 = \min(r_1, r_2)$.

Similarly, we have the following:

**THEOREM 3.9.** Let $f$ and $g \in V_\delta[A,B]$ and, for $\alpha$, $m$ positive integers, let $F$ be defined by (3.4).

Then $F \in C\left(\frac{1-A}{1-B}\right)$ for $|z| < r_0$, where $r_0$ is as given in Theorem 3.8.

**THEOREM 3.10.** Let $g \in V_\delta[A,B]$ and $\frac{f(z)}{g(z)} \in P[A,B]$ and let $F$ be defined by
\[ F(z) = \frac{m+1}{z^m} \int_0^z t^{m-1} f(t) \, dt, \]

where \( m \) is any positive integer. Then there exists a function \( G \) such that

\[ \frac{F'(z)}{G'(z)} \in \mathcal{P}\left(\frac{1-A}{1-B}\right), \quad G \in \mathcal{C}\left(\frac{1-A}{1-B}\right) \]

for \( |z| < r_i \), where \( r_i \) is given by (3.2).

**PROOF.** Let

\[ G(z) = \frac{m+1}{z^m} \int_0^z t^{m-1} g(t) \, dt. \]

Then, by Theorem 3.7 with \( \alpha = 1, \ G \in \mathcal{C}\left(\frac{1-A}{1-B}\right) \) for \( |z| < r_i \) and \( r_i \) is defined by (3.2). Now

\[
\begin{align*}
\frac{F'(z)}{G'(z)} &= \frac{z^m f(z) - m \left( \int_0^z t^{m-1} f(t) \, dt \right)}{z^m g(z) - m \left( \int_0^z t^{m-1} g(t) \, dt \right)} \\
&= \frac{\int_0^z t^{m-1} f(t) \, dt}{\int_0^z t^{m-1} g(t) \, dt} = \frac{N(z)}{D(z)}
\end{align*}
\]

Also

\[
\frac{N'(z)}{D'(z)} = \frac{f'(z)}{g'(z)} \in \mathcal{E}[A, B] \quad \text{for} \quad |z| < r_i.
\]

Thus, by Lemma 2.2, we have \( \frac{N(z)}{D(z)} \in \mathcal{E}[A, B] \subset \mathcal{P}\left(\frac{1-A}{1-B}\right) \) for \( |z| < r_i \) and this proves our result.

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