AN INVERSE EIGENVALUE PROBLEM FOR AN ARBITRARY MULTIPLY CONNECTED BOUNDED REGION IN $\mathbb{R}^2$

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ABSTRACT. The basic problem is to determine the geometry of an arbitrary multiply connected bounded region in $\mathbb{R}^2$ together with the mixed boundary conditions, from the complete knowledge of the eigenvalues $\{\lambda_j\}_{j=1}^\infty$ for the Laplace operator, using the asymptotic expansion of the spectral function $\theta(t) = \sum_{j=1}^\infty \exp(-t\lambda_j)$ as $t \to 0$.

KEY WORDS AND PHRASES. Inverse problem, Laplace’s operator, eigenvalue problem, spectral function.

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1. INTRODUCTION.

The underlying problem is to deduce the precise shape of a membrane from the complete knowledge of the eigenvalues $\{\lambda_j\}_{j=1}^\infty$ for the Laplace operator $\Delta_2 = \sum_{i=1}^2 \left( \frac{\partial}{\partial x_i} \right)^2$ in the $x^1x^2$-plane.

Let $\Omega \subseteq \mathbb{R}^2$ be a simply connected bounded domain with a smooth boundary $\partial \Omega$. Consider the Neumann/Dirichlet problem

\begin{align*}
(\Delta_2 + \lambda)u &= 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n} &= 0 \quad \text{or } u = 0 \quad \text{on } \partial \Omega,
\end{align*}

where $\frac{\partial}{\partial n}$ denotes differentiation along the inward pointing normal to $\partial \Omega$ and $u \in C^2(\Omega) \cap C(\bar{\Omega})$. Denote its eigenvalues, counted according to multiplicity, by

\begin{equation}
0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_j \leq \ldots \to \infty \quad \text{as } j \to \infty.
\end{equation}

The problem of determining the geometry of $\Omega$ has been investigated by Pleijel [1], Kac [2], McKean and Singer [3], Stewartson and Waechter [4], Smith [5], Sleeman and Zayed [6,7], Gottlieb [8], Greiner [9], Zayed [10-13] and the references given there, using the asymptotic expansion of the trace function

\begin{equation}
\theta(t) = \text{tr}[\exp(-t\Delta_2)] = \sum_{j=1}^\infty \exp(-t\lambda_j) \quad \text{as } t \to 0.
\end{equation}

It has been shown that, in the case of Neumann boundary conditions (N.b.c.):
\[ \theta(t) = \left| \frac{\Omega}{4\pi t} + \frac{1}{8(\pi t)^{1/2}} \sum_{i=1}^{k} L_i - \frac{1}{256} \left( \frac{t}{\pi} \right)^{1/2} \int_{\partial\Omega} k^2(\sigma) d\sigma + o(t) \right| \quad \text{as} \quad t \to 0, \] (1.5)

while, in the case of Dirichlet boundary conditions (D.b.c.):

\[ \theta(t) = \left| \frac{\Omega}{4\pi t} - \frac{1}{8(\pi t)^{1/2}} + \frac{a_0}{256} \left( \frac{t}{\pi} \right)^{1/2} \int_{\partial\Omega} k^2(\sigma) d\sigma + o(t) \right| \quad \text{as} \quad t \to 0, \] (1.6)

In these formulae, \(|\Omega|\) is the area of \(\Omega\), \(|\partial\Omega|\) is the total length of \(\partial\Omega\) and \(k(\alpha)\) is the curvature of \(\partial\Omega\). The constant term \(a_0\) has geometric significance, e.g., if \(\Omega\) is smooth and convex, then \(a_0 = \frac{1}{6}\) and if \(\Omega\) is permitted to have a finite number of smooth convex holes "H", then \(a_0 = \frac{1}{6}(1 - H)\).

The object of this paper is to discuss the following more general inverse problem: Let \(\Omega\) be an arbitrary multiply connected bounded region in \(\mathbb{R}^2\) which is surrounded internally by simply connected bounded domains \(\partial\Omega_i\), \(i = 1, \ldots, m-1\) and externally by a simply connected bounded domain \(\partial\Omega_m\) with a smooth boundary \(\partial\Omega_m\). Suppose that the eigenvalues (1.3) are given for the eigenvalue equation

\[ (\Delta_2 + \lambda)u = 0 \quad \text{in} \quad \Omega, \] (1.7)
together with one of the following mixed boundary conditions:

\[ \frac{\partial u}{\partial n_i} = 0 \quad \text{on} \quad \partial\Omega_i, \quad i = 1, \ldots, k \quad \text{and} \quad u = 0 \quad \text{on} \quad \partial\Omega_i, \quad i = k + 1, \ldots, m, \] (1.8)

\[ u = 0 \quad \text{on} \quad \partial\Omega_i, \quad i = 1, \ldots, k \quad \text{and} \quad \frac{\partial u}{\partial n_i} = 0 \quad \text{on} \quad \partial\Omega_i, \quad i = k + 1, \ldots, m, \] (1.9)

where \(\frac{\partial}{\partial n_i}\) denote differentiations along the inward pointing normals to the boundaries \(\partial\Omega_i, i = 1, \ldots, m\), respectively.

The basic problem is to determine the geometry of \(\Omega\) from the asymptotic expansion of the spectral function (1.4) for small positive \(t\).

Note that problems (1.7)-(1.9) have been investigated recently by Zayed [11] in the special case where \(\Omega\) is an arbitrary doubly connected bounded region (i.e., \(m=2\)).

2. STATEMENT OF OUR RESULTS.

Suppose that the boundaries \(\partial\Omega_i, i = 1, \ldots, m\) are given locally by the equations \(x^* = y^*(\sigma_i), n = 1, 2\) in which \(\sigma_i, i = 1, \ldots, m\) are the arc-lengths of the counterclockwise oriented boundaries \(\partial\Omega_i\) and \(y^*(\sigma_i) \in C^*(\partial\Omega_i)\). Let \(L_i\) and \(k_i(\sigma_i)\) be the lengths and the curvatures of \(\partial\Omega_i, i = 1, \ldots, m\) respectively. Then, the results of our main problem (1.7)-(1.9) can be summarized in the following cases:

CASE 1. (N.b.c. on \(\partial\Omega_i, i = 1, \ldots, k\) and D.b.c. on \(\partial\Omega_i, i = k + 1, \ldots, m\))

\[ \theta(t) = \left| \frac{\Omega}{4\pi t} + \frac{1}{8(\pi t)^{1/2}} \left( \sum_{i=1}^{k} L_i - \sum_{i=k+1}^{m} L_i \right) + \frac{1}{6}(2 - m) \right| + \frac{1}{256} \left( \frac{t}{\pi} \right)^{1/2} \left( \sum_{i=1}^{k} k_i^2(\sigma_i) d\sigma_i + \sum_{i=k+1}^{m} k_i^2(\sigma_i) d\sigma_i \right) + o(t) \quad \text{as} \quad t \to 0. \] (2.1)

CASE 2. (D.b.c. on \(\partial\Omega_i, i = 1, \ldots, k\) and N.b.c. on \(\partial\Omega_i, i = k + 1, \ldots, m\))

In this case the asymptotic expansion of \(\theta(t)\) as \(t \to 0\) has the same form (2.1) with the interchanges \(\partial\Omega_i, i = 1, \ldots, k \leftrightarrow \partial\Omega_i, i = k + 1, \ldots, m\).
With reference to formulae (1.4), (1.5) and to articles [6], [11], [12] the asymptotic expansion (2.1) may be interpreted as follows:

(i) $\Omega$ is an arbitrary multiply connected bounded region in $\mathbb{R}^2$ and we have the mixed boundary conditions (1.8) or (1.9) as indicated in the specifications of the two respective cases.

(ii) For the first four terms, $\Omega$ is an arbitrary multiply connected bounded region in $\mathbb{R}^2$ of area $|\Omega|$.

In case 1, it has $H = (m - 1)$ holes, the boundaries $\partial \Omega_i$, $i = 1, \ldots, k$ are of lengths $\sum_{i=1}^k L_i$ and of curvatures $k_i(\sigma_i)$, $i = 1, \ldots, m$ together with Neumann boundary conditions, while the boundaries $\partial \Omega_i$, $i = k + 1, \ldots, m$ are of lengths $\sum_{i=k+1}^m L_i$ and of curvatures $k_i(\sigma_i)$, $i = k + 1, \ldots, m$ together with Dirichlet boundary conditions, provided $H$ is an integer.

We close this section with the following remarks:

**REMARK 2.1.** On setting $k = 0$ in formula (2.1) with the usual definition that $\sum_{i=1}^0$ is zero, we obtain the results of Dirichlet boundary conditions on $\partial \Omega_i$, $i = 1, \ldots, m$.

**REMARK 2.2.** On setting $k = m$ in formula (2.1) with the usual definition that $\sum_{i=m+1}^m$ is zero, we obtain the results of Neumann boundary conditions on $\partial \Omega_i$, $i = 1, \ldots, m$.

3. FORMULATION OF THE MATHEMATICAL PROBLEM

It is easy to show that the spectral function (1.4) associated with problems (1.7)-(1.9) is given by

$$\theta(t) = \int_{\Omega} G\left(x_{-1}^{x_{-2}}, t\right) dx, \quad (3.1)$$

where $G\left(x_{-1}^{x_{-2}}, t\right)$ is Green's function for the heat equation

$$\left(\Delta - \frac{\partial}{\partial t}\right) u = 0, \quad (3.2)$$

subject to the mixed boundary conditions (1.8) or (1.9) and the initial condition

$$\lim_{t \to 0} G\left(x_{-1}^{x_{-2}}, t\right) = \delta\left(x_{-1}^{x_{-2}}\right), \quad (3.3)$$

where $\delta\left(x_{-1}^{x_{-2}}\right)$ is the Dirac delta function located at the source point $x_{-1}^{x_{-2}}$. Let us write

$$G\left(x_{-1}^{x_{-2}}, x_{-3}^{x_{-4}}, t\right) = G_0\left(x_{-1}^{x_{-2}}, x_{-3}^{x_{-4}}, t\right) + \chi\left(x_{-1}^{x_{-2}}, x_{-3}^{x_{-4}}, t\right), \quad (3.4)$$

where

$$G_0\left(x_{-1}^{x_{-2}}, x_{-3}^{x_{-4}}, t\right) = (4\pi t)^{-\frac{1}{2}} \exp \left\{ -\frac{|x_{-1}^{x_{-2}}|^2}{4t} \right\}, \quad (3.5)$$

is the "fundamental solution" of the heat equation (3.2), while $\chi\left(x_{-1}^{x_{-2}}, x_{-3}^{x_{-4}}, t\right)$ is the "regular solution" chosen so that $G\left(x_{-1}^{x_{-2}}, x_{-3}^{x_{-4}}, t\right)$ satisfies the mixed boundary conditions (1.8) or (1.9).

On setting $x_{-1}^{x_{-2}} = x_{-3}^{x_{-4}} = x$ we find that
The problem now is to determine the asymptotic expansion of $K(t)$ for small positive $t$. In what follows we shall use Laplace transforms with respect to $t$, and use $s^2$ as the Laplace transform parameter; thus we define

$$
G(x_1, x_2; s) = \int_0^\infty e^{-s^2} G(x_1, x_2; t) dt.
$$

An application of the Laplace transform to the heat equation (3.2) shows that $G(x_1, x_2; s)$ satisfies the membrane equation

$$(\Delta - s^2)G(x_1, x_2; s) = -\delta(x_1 - x_2) \quad \text{in} \quad \Omega,
$$

(3.9)

The asymptotic expansion of $K(t)$ for small positive $t$, may then be deduced directly from the asymptotic expansion of $G(s^2)$ for large positive $s$, where

$$
G(s^2) = \int_\Omega \int_\Omega \left( x_1, x_2; s^2 \right) dx_1.
$$

(3.10)

4. CONSTRUCTION OF GREEN'S FUNCTION.

It is well known [6] that the membrane equation (3.9) has the fundamental solution

$$
G_0(x_1, x_2; s) = \frac{1}{2\pi} K_0(s r_{x_1 x_2})
$$

(4.1)

where $r_{x_1 x_2} = \left| x_1 - x_2 \right|$ is the distance between the points $x_1 = (x_1^1, x_1^2)$ and $x_2 = (x_2^1, x_2^2)$ of the region $\Omega$ while $K_0$ is the modified Bessel function of the second kind and of zero order. The existence of this solution enables us to construct integral equations for $G(x_1, x_2; s^2)$ satisfying the mixed boundary conditions (1.8) or (1.9). Therefore, Green's theorem gives:

CASE 1. (N.b.c. on $\partial \Omega_i, i = 1, \ldots, k$ and D.b.c. on $\partial \Omega_i, i = k + 1, \ldots, m$)

$$
G(x_1, x_2; s^2) = \frac{1}{2\pi} K_0(s r_{x_1 x_2}) + \sum_{i=1}^k \int_{\Omega_i} \frac{\partial}{\partial n_{x_i}} K_0(s r_{x_i y}) dy + \sum_{i=k+1}^m \int_{\Omega_i} \frac{\partial}{\partial n_{y}} G(x_1, y; s^2) K_0(s r_{y_2}) dy.
$$

(4.2)

CASE 2. (D.b.c. on $\partial \Omega_i, i = 1, \ldots, k$ and N.b.c. on $\partial \Omega_i, i = k + 1, \ldots, m$)

In this case Green's function $G(x_1, x_2; s^2)$ has the same form (4.2) with the interchanges $\partial \Omega_i, i = 1, \ldots, k \leftrightarrow \partial \Omega_i, i = k + 1, \ldots, m$. 

On applying the iteration method (see [11], [12]) to the integral equation (4.2), we obtain Green's function $G(x_1, x_2; s^2)$ which has the regular part:

$$
G(x_1, x_2; s^2) = \frac{1}{2\pi^2} \sum_{n=1}^\infty \int_{\Omega_i} K_0\left(\frac{sr_{1y}}{y_2}\right) \frac{\partial}{\partial n_{1y}} K_0\left(\frac{sr_{2y}}{y_2}\right) dy
$$

$$
\quad + \frac{1}{2\pi^2} \sum_{n=1}^\infty \int_{\Omega_i} \frac{\partial}{\partial n_{1y}} K_0\left(\frac{sr_{1y}}{y_2}\right) K_0\left(\frac{sr_{2y}}{y_2}\right) dy
$$

$$
\quad + \frac{1}{2\pi^2} \sum_{n=1}^\infty \int_{\Omega_i} \frac{\partial}{\partial n_{1y}} K_0\left(\frac{sr_{1y}}{y_2}\right) M_1\left(y, y'\right) \frac{\partial}{\partial n_{2y}} K_0\left(\frac{sr_{2y}}{y_2}\right) dy dy'
$$

$$
\quad + \frac{1}{2\pi^2} \sum_{n=1}^\infty \int_{\Omega_i} \frac{\partial}{\partial n_{1y}} K_0\left(\frac{sr_{1y}}{y_2}\right) M_1\left(y, y'\right) K_0\left(\frac{sr_{2y}}{y_2}\right) dy dy'
$$

$$
\quad + \frac{1}{2\pi^2} \sum_{n=1}^\infty \int_{\Omega_i} \frac{\partial}{\partial n_{1y}} K_0\left(\frac{sr_{1y}}{y_2}\right) L_1\left(y, y'\right) \frac{\partial}{\partial n_{2y}} K_0\left(\frac{sr_{2y}}{y_2}\right) dy dy'
$$

$$
\quad + \frac{1}{2\pi^2} \sum_{n=1}^\infty \int_{\Omega_i} \frac{\partial}{\partial n_{1y}} K_0\left(\frac{sr_{1y}}{y_2}\right) L_1\left(y, y'\right) K_0\left(\frac{sr_{2y}}{y_2}\right) dy dy',
$$

where

$$
M_1\left(y, y'\right) = \sum_{n=0}^\infty K^{(y)}\left(y, y'\right),
$$

$$
M_1^{(y)}\left(y, y'\right) = \sum_{n=0}^\infty K^{(y)}\left(y, y'\right),
$$

$$
L_1\left(y, y'\right) = \sum_{n=0}^\infty K^{(y)}\left(y, y'\right),
$$

$$
L_1^{(y)}\left(y, y'\right) = \sum_{n=0}^\infty K^{(y)}\left(y, y'\right),
$$

$$
K_1\left(y, y'\right) = \frac{1}{\pi} \frac{\partial}{\partial n_{1y}} K_0\left(\frac{sr_{1y}}{y_2}\right),
$$

$$
K_1^{(y)}\left(y, y'\right) = \frac{1}{\pi} \frac{\partial}{\partial n_{1y}} K_0\left(\frac{sr_{1y}}{y_2}\right),
$$

$$
K_1^{(y)}\left(y, y'\right) = \frac{1}{\pi} K_0\left(\frac{sr_{1y}}{y_2}\right),
$$

$$
K_1^{(y)}\left(y, y'\right) = \frac{1}{\pi} K_0\left(\frac{sr_{1y}}{y_2}\right),
$$

and

$$
K_1^{(y)}\left(y, y'\right) = \frac{1}{\pi} \frac{\partial^2}{\partial n_{1y} \partial n_{1y}} K_0\left(\frac{sr_{1y}}{y_2}\right).
$$

In the same way, we can show that in case 2 Green's function $G(x_1, x_2; s^2)$ has a regular part of the same form (4.3) with the interchanges $\partial \Omega_i, i = 1, \ldots, k \leftrightarrow \partial \Omega_i, i = k + 1, \ldots, m$. 
On the basis of (4.3) the function $\chi(x_1, x_2; s^2)$ will be estimated for large values of $s$. The case when $x_1$ and $x_2$ lie in the neighborhoods of $\partial \Omega_i, i = 1, \ldots, m$ is particularly interesting. For this case, we need to use the following coordinates.

5. COORDINATES IN THE NEIGHBORHOODS OF $\partial \Omega_i, i = 1, \ldots, m$.

Let $n_i, i = 1, \ldots, m$ be the minimum distances from a point $x = (x_1, x_2)$ of the region $\Omega$ to the boundaries $\partial \Omega_i, i = 1, \ldots, m$ respectively. Let $n_i(\sigma_i), i = 1, \ldots, m$ denote the inward drawn unit normals to $\partial \Omega_i, i = 1, \ldots, m$ respectively. We note that the coordinates in the neighborhood of $\partial \Omega_i, i = k + 1, \ldots, m$ and its diagrams (see [11]) are in the same form as in section 5.1 of [11] with the interchanges $\sigma_2 \leftrightarrow \sigma_1, n_2 \leftrightarrow n_1, h_2 \leftrightarrow h_1, I_2 \leftrightarrow I_1, D(I_2) \leftrightarrow D(I_1)$ and $\delta_2 \leftrightarrow \delta_1, i = k + 1, \ldots, m$. Thus, we have the same formulae (5.1.1)-(5.1.5) of section 5.1 in [11] with the interchanges $n_2 \leftrightarrow n_1, n_i(\sigma_2) \leftrightarrow n_i(\sigma_1), t_i(\sigma_2) \leftrightarrow t_i(\sigma_1), k_i(\sigma_2) \leftrightarrow k_i(\sigma_1), i = k + 1, \ldots, m$.

Similarly, the coordinates in the neighborhood of $\partial \Omega_2, i = 1, \ldots, k$ and its diagrams (see [11]) are similar to those obtained in section 5.2 of [11] with the interchanges $\sigma_2 \leftrightarrow \sigma_1, n_2 \leftrightarrow n_1, h_2 \leftrightarrow h_1, I_2 \leftrightarrow I_1, D(I_2) \leftrightarrow D(I_1)$ and $\delta_2 \leftrightarrow \delta_1, i = 1, \ldots, k$. Thus, we have the same formulae (5.2.1)-(5.2.5) of section 5.2 in [11] with the interchanges $n_1 \leftrightarrow n_2, n_i(\sigma_1) \leftrightarrow n_i(\sigma_2), t_i(\sigma_1) \leftrightarrow t_i(\sigma_2)$ and $k_i(\sigma_1) \leftrightarrow k_i(\sigma_2), i = 1, \ldots, k$.

6. SOME LOCAL EXPANSIONS.

It now follows that the local expansions of the functions

$$K_i\left(\frac{\partial}{\partial n_{i}}, s_{xx}\right), \quad \frac{\partial}{\partial n_{i}} K_i\left(\frac{\partial}{\partial n_{i}}, s_{xx}\right), \quad i = 1, \ldots, m$$

when the distance between $x$ and $y$ is small, are very similar to those obtained in section 6 of [11]. Consequently, for $i = 1, \ldots, k, k + 1, \ldots, m$, the local behavior of the following kernels:

$$K_i\left(y', y\right), \quad K_i\left(y', y\right), \quad i = 1, \ldots, m$$

when the distance between $y$ and $y'$ is small, follows directly from the knowledge of the local expansions of (6.1).

DEFINITION 1. Let $\xi_1$ and $\xi_2$ be points in the upper half-plane $\xi^2 > 0$, then we define

$$\tilde{\rho}_{12} = \sqrt{(\xi_1 - \xi_2)^2 + (\xi_1^2 + \xi_2^2)^2}.$$  
(6.4)

An $e^{\lambda\tilde{\rho}_{12}}\left(\xi_1, \xi_2; s\right)$-function is defined for points $\xi_1$ and $\xi_2$ belong to sufficiently small domains $D(\xi_i)$ except when $\xi_1 = \xi_2 \in I_i, i = 1, \ldots, m$ and $\lambda$ is called the degree of this function. For every positive integer $\Lambda$ it has the local expansion (see [11]):
where \( \sum' \) denotes a sum of a finite number of terms in which \( f(\xi_l) \) is an infinitely differentiable function.

In this expansion, \( P_1, P_2, l, m \) are integers, where \( P_1 \geq 0, P_2 \geq 0, l \geq 0, \lambda = \min(P_1 + P_2 - q), q = l + m \) and the minimum is taken over all terms which occur in the summation \( \sum' \). The remainder \( R^A(\xi_1, \xi_2; s) \) has continuous derivatives of order \( d \leq A \) satisfying

\[
D^d R^A(\xi_1, \xi_2; s) = 0(s^{-\lambda_1}e^{-\lambda_1 s}) \quad \text{as} \quad s \to \infty,
\]

where \( A \) is a positive constant.

Thus, using methods similar to those obtained in section 7 of [11], we can show that the functions (6.1) are \( e^\lambda \)-functions with degrees \( \lambda = 0, -1 \) respectively. Consequently, the functions (6.2) are \( e^\lambda \)-functions with degrees \( \lambda = 0, -1 \), while the functions (6.3) are \( e^\lambda \)-functions with degrees \( \lambda = 0, 1 \) respectively.

**DEFINITION 2.** If \( x_1 \) and \( x_2 \) are points in large domains \( \Omega + \partial \Omega_i, i = 1, \ldots, k, k + 1, \ldots, m \), then we define

\[
f_{12} = \min_{\xi} \left( r_{x_1} + r_{y_2} \right) \quad \text{if} \quad y \in \partial \Omega_i, \quad i = 1, \ldots, k,
\]

and

\[
f_{21} = \min_{\xi} \left( r_{y_2} + r_{x_1} \right) \quad \text{if} \quad y \in \partial \Omega_i, \quad i = k + 1, \ldots, m.
\]

An \( E^{\lambda}(x_1, x_2; s) \)-function is defined and infinitely differentiable with respect to \( x_1 \) and \( x_2 \) when these points belong to large domains \( \Omega + \partial \Omega_i \) except when \( x_1 = x_2 \in \partial \Omega_i, \quad i = 1, \ldots, m \). Thus, the \( E^{\lambda} \)-function has a similar local expansion of the \( e^\lambda \)-function (see [6], [11]).

By the help of section 8 in [11], it is easily seen that formula (4.3) is an \( E^{\lambda}(x_1, x_2; s) \)-function and consequently

\[
\overline{G}(x_1, x_2; s^2) = \sum_{i=1}^{m} O \left( 1 + \left| \log s \xi_i \right| \right) e^{-A s_{12}^2}.
\]

(6.7)

which is valid for \( s \to \infty \), where \( A_i, i = 1, \ldots, m \) are positive constants.

Formula (6.7) shows \( \overline{G}(x_1, x_2; s^2) \) is exponentially small for \( s \to \infty \).

**7. THE ASYMPTOTIC BEHAVIOR OF \( \overline{\chi}(x_1, x_2; s^2) \).**

With reference to sections 7 and 9 in [11], if the \( e^\lambda \)-expansions of the functions (6.1)-(6.3) are introduced into (4.3) and if we use formulae similar to (7.4) and (7.10) of section 7 in [11], we obtain the following local behavior of \( \overline{\chi}(x_1, x_2; s^2) \) as \( s \to \infty \) which is valid when \( \xi_{12} \) and \( \hat{R}_{12} \) are small:

\[
\overline{\chi}(x_1, x_2; s^2) = \sum_{i=1}^{m} \overline{\chi}_{i}(x_1, x_2; s^2),
\]

(7.1)
where, if \( x_1 \) and \( x_2 \) belong to sufficiently small domains \( D(l_i), i = 1, \ldots, k, k + 1, \ldots, m \), then

\[
\mathcal{K}_i(x_1, x_2; s^2) = -\frac{1}{2\pi} K_0(s\tilde{\rho}_{12}) + O\{s^{-1}\exp(-A_i s\tilde{\rho}_{12})\}. \tag{7.2}
\]

When \( \tilde{r}_{12} \geq \delta_i > 0, i = 1, \ldots, k \) and \( \tilde{R}_{12} \geq \delta_i > 0, i = k + 1, \ldots, m \) the function \( \mathcal{X}_i(x_1, x_2; s^2) \) is of order \( O\{\exp(-c s)\} \) as \( s \to \infty, c > 0 \). Thus, since \( \lim_{\tilde{r}_{12} \to 0} \tilde{r}_{12} \lim_{\tilde{R}_{12} \to 0} \tilde{R}_{12} = 1 \), then if \( x_1 \) and \( x_2 \) belong to large domains \( \Omega + \partial\Omega_i, i = 1, \ldots, k \), we deduce for \( s \to \infty \) that

\[
\mathcal{K}_i(x_1, x_2; s^2) = -\frac{1}{2\pi} K_0(s\tilde{r}_{12}) + O\{s^{-1}\exp(-A_i s\tilde{r}_{12})\}, \tag{7.3}
\]

while, if \( x_1 \) and \( x_2 \) belong to large domains \( \Omega + \partial\Omega_i, i = k + 1, \ldots, m \), we deduce for \( s \to \infty \) that

\[
\mathcal{K}_i(x_1, x_2; s^2) = -\frac{1}{2\pi} K_0(s\tilde{R}_{12}) + O\{s^{-1}\exp(-A_i s\tilde{R}_{12})\}. \tag{7.4}
\]

8. CONSTRUCTION OF OUR RESULTS.

Since for \( \xi_i^2 \geq h_i > 0, i = 1, \ldots, k, k + 1, \ldots, m \), the functions \( \mathcal{X}_i(x_1, x_2; s^2) \) are of order \( O\{\exp(-2sA_i h_i)\} \), the integral of the function \( \mathcal{X}_i(x_1, x_2; s^2) \) over the region \( \Omega \) can be approximated in the following way (see (3.10)):

\[
\mathcal{K}(s^2) = \sum_{i=1}^{m} \int_{\xi_i^2 = 0}^{\tilde{r}_i} \int_{\xi_i^2 = 0}^{\tilde{R}_i} \mathcal{X}_i(x_1, x_2; s^2) \{1 - k_i(\xi_i^2)\xi_i^2\} d\xi_i^2 d\xi_i^2
\]

\[
- \frac{1}{2} \sum_{i=1}^{m} \int_{\xi_i^2 = 0}^{\tilde{r}_i} \int_{\xi_i^2 = 0}^{\tilde{R}_i} \mathcal{X}_i(x_1, x_2; s^2) \{1 + k_i(\xi_i^2)\xi_i^2\} d\xi_i^2 d\xi_i^2
\]

\[
+ \sum_{i=1}^{m} O\{\exp(-2sA_i h_i)\} \quad \text{as} \quad s \to \infty. \tag{8.1}
\]

If the \( e^x \)-expansions of \( \mathcal{X}_i(x_1, x_2; s^2) \), \( i = 1, \ldots, k, k + 1, \ldots, m \), are introduced into (8.1), one obtains an asymptotic series of the form:

\[
\mathcal{K}(s^2) = \sum_{n=1}^{j} a_n s^n + O(s^{j+1}) \quad \text{as} \quad s \to \infty, \tag{8.2}
\]

where the coefficients \( a_n \) are calculated from the \( e^x \)-expansions by the help of formula (10.3) of section 10 in [11].

Now, the first three coefficients \( a_1, a_2, a_3 \) take the forms:
\begin{equation}
 a_1 = \frac{1}{8} \left( \sum_{i=1}^{4} L_i - \sum_{i=1}^{8} L_i \right),
 \end{equation}
\begin{equation}
 a_2 = \frac{1}{6} (2 - m),
 \end{equation}
\begin{equation}
 a_3 = \frac{1}{512} \left[ 7 \sum_{i=1}^{4} \int_{\Omega_i} k_i^2(\omega) d\omega + \sum_{i=1}^{8} \int_{\Omega_i} k_i^2(\omega) d\omega \right].
 \end{equation}

On inserting (8.3) into (8.2) and inverting Laplace transforms and using (3.6) we arrive at our result (2.1).

REFERENCES


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