ON POLYNOMIAL EXPANSION OF MULTIVALENT FUNCTIONS

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ABSTRACT. Coefficient bounds for mean p-valent functions, whose expansion in an ellipse has a Jacobi polynomial series, are given in this paper.

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1. INTRODUCTION.

Let \( E_0 = \{ z = \cosh(s + i\tau), \ 0 < \tau < 2\pi, \ s = \tanh^{-1}(b/a), \ a > b > 0 \} \) be a fixed ellipse whose foci are \( \pm 1 \). Let also \( r_0 = a + b \) be the sum of the semi-axis of \( E_0 \). It is known (Szegö [1], Theorem 9.1.1, see also p. 245) that a function \( f(z) \) which is regular in \( \text{Int}(E_0) \) (this means the interior of \( E_0 \)) has an expansion of the form

\[
f(z) = \sum_{n=0}^{\infty} a_n P_n(z)
\]

where here and throughout this paper \( a_n \) are

\[
W(R,f) = (1/R) \int_0^R \int_0^{2\pi} n(\phi, f, \text{Int}(E_0)) \, d\phi \, d\theta < R^2
\]

where \( 0 < R < \infty \) and \( n(\phi, f, \text{Int}(E_0)) \) denotes the number of roots of the equation \( f(z) = w \) in \( \text{Interior} \ E_0 \), multiplicity being take into account.

We first recall from [2]:

THEOREM A. Let \( f(z) \) be mean p-valent in \( \text{Int}(E_0) \). Then for \( z = \cosh(s + i\tau) \), \( \exp(s) = r \) and \( 1 < r < r_0 \), we have

\[
|f(z)| = O(1) \left(1 - r/r_0\right)^{-2p}
\]

where \( O(1) \) depends on \( a, b \) and \( f \) only.
THEOREM B. Let $f(z)$ be mean $p$-valent in $\text{Int}(E)$ and $M(r,f) < C(1-r/r_o)^{-\gamma}$
where $c, \gamma > 0$ and $M(r,f) = \max\{|f(z)|: z \in \text{Int}(E_o)\}$. Set $z = \cosh(s+i\tau)$, $\exp(s) = r$, $1 < r < r_o$ and

$$I_1(r,f') = (1/2\pi) \int_0^{2\pi} |f'(\cosh(s+i\tau))||\sinh(s+i\tau)|dt.$$ 

Then as $r \to r_o$ we have

$$I_1(r,f') = \begin{cases} 0(1) (1-r/r_o)^{-\gamma}, & (\gamma > 1/2), \\ 0(1) (1-r/r_o)^{-1/2}\log(1/(1-r/r_o)), & (\gamma = 1/2), \\ o(1) (1-r/r_o)^{-1/2}, & (\gamma < 1/2), \end{cases}$$

where $0(1)$ and $o(1)$ depend on $a, b, \gamma$ and $f$ only.

PROOF OF THEOREM B. Using Schwarz's inequality we have

$$I_1(r,f') \leq [(1/2\pi) \int_0^{2\pi} |f'(\cosh(s+i\tau))|^2|f(\cosh(s+i\tau))|^2dt]^{1/2} \leq 1/2\pi \int_0^{2\pi} |f(\cosh(s+i\tau))|^2|\sinh(s+i\tau)|^2dt$$

where $0 < \lambda < 2$. Theorem B now follows in the same way as estimating inequality (14) of [2] by using [2, Lemmas 3 and 4].

We now need a suitable coefficient formula.

LEMMA 1.1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be regular in $\text{Int}(E)$ and

$$E = \{z = \cosh(s+i\tau), 0 < \tau < 2\pi\}.$$ 

Then for a fixed $s$ so that $0 < s < s_o$ we have

$$a_n = (K_n^{(0)}(a, b))/h_n^{(0)}(a, b)(1/2\pi) \int_{E} f(z) dz, \quad (n > 0),$$

$$\frac{1}{2}(n+a+1) a_n = (K_n^{(0)}(a+1, b+1))/h_n^{(0)}(a+1, b+1)(1/2\pi) \int_{E} f(z) dz, \quad (n > 1)$$

where

$$K_n^{(0)}(a, b) = 2^{n+a+b+1} \Gamma(n+a+1)\Gamma(n+b+1)/\Gamma(2n+a+b+2)$$

and

$$h_n^{(0)}(a, b) = 2^{a+b+1} \Gamma(n+a+1)\Gamma(n+b+1)/(2n+a+b+1)\Gamma(n+1)\Gamma(n+a+b+1).$$

We note here, using Stirling's formula from Titchmarsh [3, p. 57], that

$$K_n^{(0)}(a, b)/h_n^{(0)}(a, b) = 0(1)n^{1/2}/2^n$$

as $n \to \infty$, where $0(1)$ depends on $a, b$ only.

PROOF OF LEMMA. We have from [1, p. 245] that

$$a_n = \{\min(\alpha, \beta)\}^{-1} \int_{E} (z-1)^{a} (z+1)^{b} \delta_n^{(a, \beta)}(z)f(z) dz$$

where $n = 0, 1, 2, \ldots$.

We now see from [1, Theorem 4.61.2], (see also Erdelyi, Magnus, Oberhettinger and Tricomi [4, p. 171], and Freud [5, p. 44] that

$$\int_{-\infty}^{\infty} (z-1)^{a} (z+1)^{b} \delta_n^{(a, \beta)}(z) dz = (1/2) \sum_{k=0}^{\infty} \frac{1}{k+1} \int_{-1}^{1} (1-t)^{a}(1+t)^{b} \delta_n^{(a, \beta)}(t) dt$$

$$= K_n^{(0)}(a, b)/2n+1,$$
where \( k_n^{(a,b)} \) is as defined above. In connection with this, see the argument used in the proof of formula (4.3.3) of [1, p.67].

Using (1.6) in (1.5) we immediately deduce (1.2).

Now differentiating (1.1) we see from equation (4.21.7) of [II that

\[
f'(z) = \sum_{n=1}^{\infty} \frac{1}{2(n+a+b+1)} a_n P_n^{(a+1,b+1)}(z).
\]

Again, as in the proof of (1.2), we deduce from this and [1, p. 245] for \( n > 1 \), that

\[
\frac{1}{2(n+a+b+1)} a_n = \left( \frac{1}{n-1} \right) \int_E \frac{f'(z) dz}{(z-1)^{n-1}(z+1)^{n-1}} P_n^{(a+1,b+1)}(z) + \frac{1}{(2\pi i)^n} \int_E \frac{f'(z) dz}{z^n}.
\]

where we have used the equation (z-1)^{a+1}(z+1)^{b+1} = k_n^{(a+1,b+1)}(z) which is deduced as in (1.6). This is equation (1.3) and the proof of the lemma is now complete.

2. MAIN THEOREM.

THEOREM 2.1. Let \( f(z) = \sum_{n=0}^{\infty} a_n P_n^{(a,b)}(z) \) be mean \( p \)-valent in \( \text{Int}(E_0) \) and

\[
M(r,f) < C(1-r/r_o)^{-\gamma} \text{ where } C, \gamma > 0 \text{ and } M(r,f) \text{ is as defined above. Then, as } n \to \infty \text{ we have}
\]

\[
|a_n| = r_n^{-\frac{1}{2}} \begin{cases} 0(1)n^{-\gamma/2}, & (\gamma < 1/2), \\ 0(1) \log n, & (\gamma = 1/2), \\ o(1), & (\gamma < 1/2), \end{cases}
\]

where \( 0(1) \) and \( o(1) \) depend on \( a,b,a_0,f \) and \( f \) only.

PROOF OF THEOREM 2.1. From (1.3) and Theorem B we deduce, using the bounds

\[
\sinh(s+t) > \sinh s, \quad |\cosh(s+t)| < \cosh s \quad \text{and} \quad (1.4), \quad \text{that}
\]

\[
\frac{1}{2(n+a+b+1)} |a_n| < \left( \frac{1}{n-1} \right) \int_E \frac{f'(z) dz}{(z-1)^{n-1}(z+1)^{n-1}} P_n^{(a+1,b+1)}(z) + \frac{1}{(2\pi i)^n} \int_E \frac{f'(z) dz}{z^n}.
\]

where we have chosen \( r = ((n-1)/n)r_o \) and provided that \( 1-n/(n-1)r_o > 0 \). This completes the proof of Theorem 2.1.

COROLLARY 2.1. Let \( f(z) = \sum_{n=0}^{\infty} a_n P_n^{(a,b)}(z) \) be mean \( p \)-valent in \( \text{Int}(E_0) \). Then, as \( n \to \infty \) we have

\[
|a_n| = r_n^{-\frac{1}{2}} \begin{cases} 0(1)n^{2p-1/2}, & (p > 1/4), \\ 0(1) \log n, & (p = 1/4), \\ o(1), & (p < 1/4), \end{cases}
\]
where \( \vartheta(1) \) and \( \vartheta(1) \) depend on \( a, b, a, \alpha, \beta, p \) and \( f \) only. In view of Theorem A, the proof of Corollary 2.1 follows by setting \( r = 2p \) in Theorem 2.1.

**COROLLARY 2.2.** Let \( f(z) = \sum_{n=0}^{\infty} a_n p_n^{(a, \beta)}(z) \) be univalent in \( \text{Int}(E_0) \). Then as \( n \to \infty \) we have

\[
|a_n| = \vartheta(1) n^{3/2} r_o^n
\]

where \( \vartheta(1) \) depends on \( a, b, a, \beta \) and \( f \) only.

This corollary follows upon setting \( p = 1 \) in Corollary 2.1.

**REMARK.** Using the formula (4.21.2) of [1] and the argument used in [2, Remark 2] we see by setting \( z = \xi \cosh s_0 \) where \( |\xi| = |\cos \tau + i \tanh s_0 \sin \tau| < 1 \) that

\[
f(\xi \cosh s_0) = \sum_{n=0}^{\infty} \frac{\Gamma(2n+\alpha+\beta+1)}{n! \Gamma(n+\alpha+\beta+1)} a_n \left( \frac{\cosh s_0}{\xi} \right)^n \left( (\xi - 1/\cosh s_0) \right)^n
\]

where

\[
p_n^{(a, \beta)}(\xi) = (\xi - 1/\cosh s_0)^n + c_1 (\xi - 1/\cosh s_0)^{n-1} + \ldots + c_n / \cosh^n s_0
\]

and

\[
a_n = \frac{\Gamma(2n+\alpha+\beta+1)a_n \cosh^n s_0 / 2^n \Gamma(n+1) \Gamma(n+\alpha+\beta+1)}{\cosh s_0}.
\]

Using this and Stirling's formula and letting \( r_o \to \infty \) we see that Theorem 2.1 and Corollaries 2.1 and 2.2 correspond to analogous results for the unit disk (see Hayman [6]).

**REFERENCES**