EXISTENCE OF SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, using a simple and classical application of the Leray-Schauder degree theory, we study the existence of solutions of the following boundary value problem for functional differential equations

\[ x''(t) + f(t, x(t), x'(t)) = 0, \quad t \in [0, T] \]
\[ x_0 + x'(0) = h \]
\[ x(T) + \beta x'(T) = n \]

where \( f \in C([0, T] \times \mathbb{R}^n, \mathbb{R}^n) \), \( h \in \mathbb{R}^n \), \( n \in \mathbb{R}^n \), and \( \alpha, \beta \) are real constants.

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1. INTRODUCTION

Let \( \mathbb{R}^n \) be the real euclidean space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( |\cdot| \). Let also, \( C_r \) be the space of all continuous functions \( x: [-r, 0] \to \mathbb{R}^n, \ r > 0 \), endowed with the sup-norm

\[ ||x|| = \sup\{|x(t)| : t \in [-r, 0]\}. \]

For every continuous function \( x: [-r, T] \to \mathbb{R}^n, \ T > 0 \) and every \( t \in [0, T] \), we denote by \( x_t \) the element of \( C_r \), defined by

\[ x_t(0) = x(t+0), \quad 0 \in [-r, 0]. \]
The main purpose of this paper is to discuss when the functional differential equation
\[ x''(t) + f(t, x(t), x'(t)) = 0, \quad t \in [0, T], \] (1.1)
admits a solution \( x \) on \([0, T']\) such that the boundary value conditions
\[ x(0) + \alpha x'(0) = h \] (1.2a)
\[ x(T) + \beta x'(T) = \eta \] (1.2b)
to be satisfied. Here, \( f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n \) is a continuous function, \( h \in \mathbb{C}_r, \eta \in \mathbb{R}^n \) and \( \alpha, \beta \) are real constants such that
\[ 0 < \alpha \leq \beta \] (1.2c)
By \( x'(0) \) and \( x'(T) \) we mean \( x'(0^-) \) and \( x(T^-) \), respectively. In the next, the boundary value problem (B.V.P.) which constitutes from the equation (1.1) and the boundary conditions (1.2a), (1.2b), (1.2c), will be mentioned briefly as B.V.P. (1.1)-(1.2).

Analogous boundary value problems for ordinary differential equations has been studied by many authors, who used the Leray-Schauder continuation theorem (see Lasota and Yorke [1], Szmanda [2], Tracpe [3] and others). Usually, in these problems the authors derive a priori estimates of solutions by using inequalities of Wirtinger and Opial type.

Our work is motivated by the recent papers of Fabry and Habets [4], Fabry [5] and Ntouyas [6]. In [6] the author generalizes the results of Fabry and Habets [4] to the functional equation (1.1) with boundary conditions
\[ x(0) = h, \quad h(0) = 0, \]
\[ x(T) = 0. \]
Here, following Fabry [5] we extend the results of Ntouyas [6].

2. MAIN RESULTS

Before stating our main results we refer some lemmas which simplify the proof of the theorem below.

**Lemma 2.1.** [4, pp 187]. Let \( X \) be a Banach space, \( A : X \to X \) be a completely continuous mapping such that \( I - A \) is one to one, and let \( \Omega \) be a bounded set such that \( 0 \in (I - A)(\Omega) \). Then the completely continuous mapping \( S : \Omega \to X \) has a fixed point in \( \Omega \) if for any \( \lambda \in (0, 1) \), the equation
\[ x = \lambda Sx + (1 - \lambda)Ax \] (2.1)
has no solution on the boundary \( \partial \Omega \) of \( \Omega \).

**Lemma 2.2.** [5, pp 133]. Let \( X : [0, T] \to \mathbb{R}^n \) be a twice differentiable function and let \( R > 0 \) be such that
\[ ||x|| \leq R. \] (2.2)
Assume that positive constants \( c, d \) exist, with \( c < 1 \), such that
\[ \langle x(t), x''(t) \rangle \leq c|x'(t)|^2 + d, \quad t \in [0, T]. \] (2.3)
Moreover, assume that positive constants \( c', d' \) exist with \( c' < (1 - c)^2/8R \) such that
\[ |\langle x'(t), x'''(t) \rangle| \leq (c'|x'(t)|^2 + d')|x'(t)|, \quad t \in [0, T]. \] (2.4)
Then there exists a number \( K \) nondepending on \( x \), such that
\[ ||x'(t)|| \leq K. \]

**Lemma 3.2.** If \( \alpha \leq 0 \leq \beta \), the B.V.P.
EXISTENCE OF SOLUTIONS OF BOUNDARY VALUE PROBLEMS

\[ x''(t) = kx(t), \quad k > 0 \]
\[ x(0) + \alpha x'(0) = 0, \quad x(T) + \beta x'(T) = 0 \]

has the unique solution \( x = 0 \).

**Proof.** The general solution of the above equation has the form
\[ x(t) = c_1 e^{-\sqrt{k}t} + c_2 e^{\sqrt{k}t}. \]

On account of the above boundary conditions, we obtain
\[ \frac{(1+\alpha \sqrt{k})(1-\beta \sqrt{k})}{(1-\alpha \sqrt{k})(1+\beta \sqrt{k})} < e^{\sqrt{k}T}. \]

Since \( e^{2\sqrt{k}T} > 1, \quad k > 0 \), the last expression is true for every \( k > 0 \), provided the left hand side is less than or equal to one. But this is clear since \( \alpha \leq \beta \leq 0 \).

The next Theorem guarantees existence of solutions for the B.V.P. (1.1)-(1.2) which are bounded by an a priori given function \( \psi \). Moreover, the first derivative of such a solution is also bounded by a constant \( \rho \) not depending on this solution.

**Theorem.** Let \( f : [0,T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a continuous function which maps bounded sets of \( [0,T] \times \mathbb{R}^n \) into bounded sets of \( \mathbb{R}^n \). Assume that \( f(t,x,x') \) is a twice continuously differentiable function such that
\[ -\psi(0) - |\alpha| \psi'(0) > |h(0)|, \quad \text{if} \quad \alpha \neq 0 \]
\[ \psi(0) > |h(0)|, \quad \text{if} \quad \alpha = 0 \]  \hspace{1cm} (2.5a)

and
\[ -\psi(T) + |\beta| \psi'(T) > |h|, \quad \text{if} \quad \beta \neq 0 \]
\[ \psi(T) > |h|, \quad \text{if} \quad \beta = 0. \]  \hspace{1cm} (2.5b)

Also, we suppose that
\[ \psi(t) \psi''(t) - u(t), \psi(t), u(t), v(t) \leq 0 \]  \hspace{1cm} (2.6)
for any \( (t,u,v) \in [0,T] \times \mathbb{R}^n \) with \( u(t) = |u(t)| \) and \( <u(t),v> = |u(t)| \psi'(t) \).

Moreover, assume that there exist positive numbers \( k_1, k_2 \) with \( k_1 < 1 \) and positive numbers \( k_1', k_2' \) such that
\[ k_1' < \frac{1}{8m} (1-k_1)^2, \quad m = \max_{t \in [0,T]} |\psi(t)| \]
\[ <u(t), f(t,u,v)> \leq k_1 |v|^2 + k_2, \]  \hspace{1cm} (2.7)
\[ <v, f(t,u,v)> \leq (k_1' |v|^2 + k_2') |v| \]  \hspace{1cm} (2.8)
for any \( (t,u,v) \in [0,T] \times \mathbb{R}^n \) with \( |u(t)| \leq \psi(t) \).

Then the problem (1.1)-(1.2) has at least one solution \( x \) such that \( |x(t)| \leq \psi(t) \), \( t \in [0,T] \) and \( |x'(t)| \leq \rho, \quad t \in [0,T] \).

**Proof.** Let \( k > 0 \) be a constant, such that \( k > \max \left\{ \frac{\psi''(t)}{\psi(t)}, \quad t \in [0,T] \right\} \) and \( x \) a solution of the equation
\[ x''(t) + \lambda f(t,x,x') = (1-\lambda)kx(t), \quad \lambda \in (0,1) \]  \hspace{1cm} (2.9)
with \( t \in [0,T] \) and \( |x(t)| \leq \psi(t) \).

Multiplying both sides of (2.9) by \( x(t) \) and using (2.7) we deduce that
\[ -<x(t),x''(t)> = \lambda <x(t), f(t,x,x') - (1-\lambda)kx(t)>^2 \]
\[ \leq \lambda (k_1 |x'(t)|^2 + k_2) \]
Similarly, condition (2.8) yields
\[\langle x'(t), x''(t) \rangle \leq (k_1|x'(t)|^2 + k_2)|x'(t)|m\]
where \(\epsilon = k_1^2 + k_2m\).

Thus, the conditions of Lemma 2.2 are fulfilled and hence there exists a number \(K\) not depending on \(x\), such that \(|x'(t)| \leq K\).

Let us now consider the Banach space \(B\) of all continuous functions \(x: [0, T] \to \mathbb{R}^n\), which are continuously differentiable on \([0, T]\), endowed with the norm
\[\|x\|_1 = \max \left\{ \sup_{t \in [0, T]} |x(t)|, \sup_{t \in [0, T]} |x'(t)| \right\} .\]

Also, for any \(x \in B\) we set
\[Sx(t) = \int_0^T G(t, s)f(s, x(s), x'(s))ds + \frac{1}{k} \left[ (T-t)h(0) + Bh(0) - at + n \right], t \in [0, T] \quad (2.10a)\]
where
\[x_s(\theta) = \begin{cases} x(s + \theta), & \text{if } \theta \geq -s \\ h(s + \theta - ax'(0)), & \text{if } \theta < -s. \end{cases} \quad (2.10b)\]

Here, \(G\) is the Green function for the B.V.P.
\[y'' = 0, \quad y(0) + ay'(0) = 0, \quad y(T) + By'(T) = 0\]
and is given by the formula
\[G(t, s) = \begin{cases} \frac{(t - T - \beta)(s - a),} {k} \quad & s \leq T \\ \frac{(t - \alpha)(s - T - \beta),} {k} \quad & t \leq s, \end{cases}\]
where \(k = T + B - \alpha \neq 0\) because of (1.2c).

Obviously, the operator \(S\) is a compact operator defined on \(B\) and taking values in \(B\).

Since the B.V.P. (1.1)-(1.2) is equivalent to (2.10a) and (2.10b), the purpose of the following proof is to show that the mapping \(S\) has a fixed point.

To this end we define an operator \(A: B \to B\), and a subset \(\Omega\) of \(B\) as follows:
\[(Ax)(t) = -\int_0^T G(t, s)k x(s)ds, \quad k \neq 0 \quad (2.11)\]
and
\[\Omega = \{x \in B : \forall t \in [0, T], \ |x(t)| < \varphi(t), \ |x'(t)| < K + 1\}, \quad (2.12)\]
where \(k \neq 0\) and \(K\) are defined as above.

It is clear that \(\Omega\) is open and bounded in \(B\) and \(A\) is a completely continuous operator.

First we prove that the operator \((I - A)x = (I - A)y\). If \(z(t) = x(t) - y(t)\) then \((I - A)z = 0\) and \(z(0) + az'(0) = 0, \ z(T) + Bz'(T) = 0\). Hence, \(z\) is a solution of the B.V.P.
\[z''(t) = k z(t), \quad z(0) + az'(0) = 0, \quad z(T) + Bz'(T) = 0.\]
EXISTENCE OF SOLUTIONS OF BOUNDARY VALUE PROBLEMS

By Lemma 2.3 the last problem has the unique solution \( z = 0 \), and consequently \( I - A \) is one to one.

Next, we show that for any \( \lambda \in [0,1] \) and \( x \in \Omega \) it is the case that

\[ x \neq \lambda sx + (1 - \lambda)Ax \]

Indeed, if there exists \( \lambda \in [0,1] \) and \( x \in \Omega \) satisfying

\[ x = \lambda sx + (1 - \lambda)Ax, \]

then the equation

\[ x''(t) + \lambda f(t, x, x'(t)) = (1 - \lambda)kx(t), \]

has a solution \( x : [0, T] \rightarrow \mathbb{R}^n \) satisfying

\[ x(0) + ax'(0) = h \]
\[ x(T) = \psi(x'(T)) = n \]
\( x \in \Omega. \)

Hence there exist \( \xi, \eta \in [0, T] \) such that either

\[ |x(\xi)| = \psi(\xi) \text{ or } |x'(\xi)| = k + 1. \]

Now, we shall prove that, in view of (2.13a), (2.138), the relations in (2.14) cannot hold. Since \( x \) is a solution of (2.9) for some \( \lambda \in [0,1] \), the computation following (2.9) show that \( |x'(t)| \leq K \) and hence \( |x'(t)| < K + 1, 0 \leq t \leq T \). Hence, the second case in (2.14) cannot hold. Thus it remains to eliminate the first possibility of (2.14). We shall prove that if \( x \in \Omega \) is a solution of (2.9), then there exists no \( \xi \in [0, T] \) such that \( |x(t)|^2 - \psi^2(t) \) reaches maximum value zero at \( t = \xi \in [0, T] \).

Assume the contrary. Then, if \( \xi \in (0, T) \), we have the following relations

\[ |x(\xi)| = \psi(\xi) \]
\[ <x(\xi), x'(\xi)> = \psi(\xi)\psi'(\xi) \]
\[ or \]
\[ <x(\xi), x'(\xi)> = \psi(\xi)\psi'(\xi) \]
\[ J = <x(0), x'(0) + |x'(\xi)|^2 - \psi(\xi)\psi'(\xi) \]
\[ \geq (1 - \lambda)(|x'(\lambda)|^2 - \psi^2(\xi) - \psi(\xi)\psi'(\xi)) \geq 0. \]

Now assume that \( x \) is a solution of (2.9). Then by (2.6), (2.15), (2.168), we obtain

\[ J = -\lambda <x(0), f(t, x, x'(t)) + (1 - \lambda)k|x(\xi)|^2 + |x'(\xi)|^2 - \psi(\xi)\psi'(\xi)^2(\xi) \]

\[ \geq (1 - \lambda)(|x'(\lambda)|^2 - \psi^2(\xi) - \psi(\xi)\psi'(\xi) + k|x(\xi)|^2) \]

\[ \geq (1 - \lambda)\psi'(\xi)(k\psi(\xi) - \psi''(\xi)). \]

Since \( k > \psi''(\xi)/\psi'(\xi) \), \( t \in (0, T) \), we get \( J > 0, \lambda \in [0,1] \), contradicting (2.17).

Next we show that \( \xi \neq T \). If \( \xi = T \) and \( g(t) = |x(t)|^2 - \psi^2(t) \) then the following must hold:

\[ g'(T) = 2<x(T), x'(T)> - 2\psi(T)\psi'(T) \geq 0 \]

and

\[ g(T) = 0. \]

Then \( |x(T)| = \psi(T) \) and \( \psi'(T) \leq |x'(T)| \). But, by the boundary condition (1.2b), we have
Hence
\[ |\beta| \leq |x'(T)| \leq |n| + \varphi(T). \]

or
\[ |\beta| \leq |n| + \varphi(T), \text{ if } \beta \neq 0 \]

or
\[ \varphi(T) \leq |n|, \text{ if } \beta = 0 \]

which contradicts (2.5b). Therefore \( \xi \neq T \) as required.

Finally, we show that \( \xi \neq 0 \). Assume on the contrary that \( \xi = 0 \). It is straightforward to see that
\[ g(0) = 0 \text{ and } g'(0) \leq 0, \]

 imply
\[ |x(0)| = \varphi(0) \text{ and } |x'(0)| \leq \varphi'(0) \]

From the boundary condition (1.2a) we obtain
\[ -\varphi(0) \leq |h(0)| + |a| \varphi'(0), \text{ if } a \neq 0 \]

or
\[ \varphi(0) \leq h(0), \text{ if } a = 0, \]

contradicting (2.5a).

Consequently, no solutions of (2.9) can belong to \( \mathcal{A} \) for \( \lambda \in [0,1) \), completing the proof of the theorem.

3. APPLICATIONS

As an application of the Theorem we consider the equation
\[ x''(t) + \ell(t,x_t)x'(t) + p(t,x_t)x(t) + q(t,x_t) = 0, \quad t \in [0,T] \]  

where \( \ell \) and \( p \) are bounded real valued functions defined on \([0,T] \times \mathbb{R}^n\) and \( q \) is also bounded \( \mathbb{R}^n \)-valued function defined on \([0,T] \times \mathbb{R}^n\).

We set
\[ \hat{\ell} = \sup_{(t,u) \in [0,T] \times \mathbb{R}^n} |\ell(t,u)|, \quad \hat{p} = \sup_{(t,u) \in [0,T] \times \mathbb{R}^n} |p(t,u)|, \quad \hat{q} = \sup_{(t,u) \in [0,T] \times \mathbb{R}^n} |q(t,u)|. \]

Then we have the following

**Proposition.** If there exists a constant \( M \),
\[ M \geq \max \{ \hat{\ell}, \hat{p}, \hat{q} \} \]

such that the inequality
\[ \varphi''(t) + M[|\varphi'(t)| + \varphi(t) + 1] \leq 0, \quad t \in [0,T] \]

has a strictly positive solution \( \varphi \), subject to the conditions (2.5a), (2.5b), then the B.V.P. (3.1)-(1.2) has at least one solution satisfying
\[ |x(t)| \leq \varphi(t), \quad t \in [0,T]. \]

Moreover, there exists \( \rho \) not depending on \( x \) with
\[ |x'(t)| \leq \rho, \quad t \in [0,T]. \]

**Proof.** It is enough to check the conditions of the theorem for the function
\[ f(t,u,v) = \ell(t,u) + p(t,u)v + q(t,u), \quad (t,u,v) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n. \]

Indeed, for any \( (t,u,v) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \), with \( |u(t)| = \varphi(t) \) and \( <u(0),v> = |u(0)||\varphi'(t)| \), we obtain
\[ <u(0),f(t,u,v) = \ell(t,u) <u(0),v> + p(t,u)|u(0)|^2 + <u(0),q(t,u)> \]
In view of (3.2), the above relation shows that (2.6) holds.

Also, for any \((t,u,v) \in [0,T] \times \mathbb{R}^p \times \mathbb{R}^n\) with \(|u(0)| \leq q(t)\) we get, obviously,

\[
<u(0), f(t,u,v) > \leq \tilde{L} q(t) + \tilde{L} \|v\| + \tilde{L} \|v\| + \tilde{q} q(t)
\]

where \(c_1 = \sup_{t \in [0,T]} (\tilde{p} \psi^2(t) + \tilde{q} q(t))\) and \(c_2 = \sup_{t \in [0,T]} (\tilde{L} \psi(t))\).

Moreover,

\[
<v, f(t,u,v) > \leq \tilde{L} \|v\| + \tilde{L} \|v\| \leq c_1 \|v\| + \tilde{L} \|v\|^2,
\]

where \(c_1^* = \sup_{t \in [0,T]} (\tilde{p} \psi(t) + \tilde{q})\). Now, if \(|v| \geq 1\) then we have \(c_1^* \|v\| + \tilde{L} \|v\|^2 \leq (c_1^* + \tilde{L}) \|v\|^2\). If \(|v| < 1\) then (2.8) follows from the inequality

\[
\tilde{L} \geq \tilde{L} \|v\| - \frac{1}{2} \|v\|^2, \text{ for each } \tilde{L} \geq 0.
\]

Indeed, we have

\[
c_1^* \tilde{L} \|v\| = c_1^* \tilde{L} \|v\|^2 + \tilde{L} \|v\| - \frac{1}{2} \|v\|^2 \leq c_1^* \tilde{L} \|v\|^2 + \tilde{L}.
\]

Hence (2.8) is satisfied for \(k_1 = \tilde{L} \) and \(k_2 = c_1^* \tilde{L}\).

**EXAMPLE.** The B.V.P.

\[
x''(t) + \frac{x(t)}{1+ \|x\|} x'(t) = 0, \quad t \in [0,1]
\]

\[x(0) = h, \quad x(1) + \beta x'(1) = \eta.
\]

has at least one solution \(x\) such that

\[|x(t)| \leq 2 - e^{-t}
\]

provided that function \(h\) and constants \(\beta\) and \(\eta\) are such that

\[|h(0)| < 1 \text{ and } |\beta| + 1 > e(2 + |\eta|).
\]

We remark that in this case \(\tilde{L} = 1\) (and hence \(M = 1\)) and (3.2) becomes \(\psi''(t) + \psi(t) \leq 0\), \(t \in [0,1]\).

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