ON THE ARENS PRODUCTS AND REFLEXIVE BANACH ALGEBRAS

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ABSTRACT. We give a characterization of reflexive Banach algebras involving the Arens product.

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1. INTRODUCTION.

Let $A$ be a semisimple Banach algebra and $A^{**}$ the second conjugate space of $A$ with the Arens product $\circ$. If $(A^{**}, \circ)$ is semisimple and it has a dense socle, then we show that the following statements are equivalent: (1) $A$ is reflexive. (2) $A^{**}$ is w.c.c. (3) $A$ is w.c.c. (4) $A$ and $A^{**}$ have the same socle. This is a generalization of a result by Duncan and Hosseinioum [1, p.319, Theorem 6(ii)]. We also show that if $(A^{**}, \circ)$ is semisimple and $A$ is l.w.c.c., then $A$ is Arens regular.

2. NOTATION AND PRELIMINARIES. Definitions not explicitly given are taken from Rickart’s book [2].

Let $A$ be a Banach algebra. Then $A^*$ and $A^{**}$ will denote the first and second conjugate spaces of $A$, and $\pi$ the canonical map of $A$ into $A^{**}$. The two Arens products on $A^{**}$ are defined in stages according to the following rules (see [3] and [4]). Let $x, y \in A, f \in A^*$, and $F, G \in A^{**}$.

Define $f \circ x$ by $(f \circ x)(y) = f(xy)$. Then $f \circ x \in A^{**}$.
Define $G \circ f$ by $(G \circ f)(x) = G(fx)$. Then $G \circ f \in A^*$.
Define $F \circ G$ by $(F \circ G)(f) = F(Gf)$. Then $F \circ G \in A^{**}$.
Define $x \circ f$ by $(x \circ f)(y) = f(yx)$. Then $x \circ f \in A^*$.
Define $f \circ F$ by $(f \circ F)(x) = F(x \circ f)$. Then $f \circ F \in A^*$.
Define $F \circ G$ by $(F \circ G)(f) = G(f \circ F)$. Then $F \circ G \in A^{**}$.

$A^{**}$ is a Banach algebra under the products $F \circ G$ and $f \circ F$ and $\pi$ is an algebra isomorphism of $A$ into $(A^{**}, \circ)$ and $(A^{**}, \circ')$. In general, $\circ$ and $\circ'$ are distinct on $A^{**}$. If they agree on $A^{**}$, then $A$ is called Arens regular.
LEMMA 2.1. Let $A$ be a Banach algebra. Then, for all $x \in A, f \in A^*$, and $F, G \in A^{**}$, we have

1. $\pi(x) o F = \pi(x) o' F$ and $F o \pi(x) = F o' \pi(x)$.
2. If $\{F_t\} \subset A^{**}$ and $F_t \rightharpoonup F$ weakly in $A^{**}$, then $F_t o G \rightharpoonup F o G$ and $G o' F_t \rightharpoonup G o' F$ weakly.

PROOF. See [3, p.842 and p. 843].

Let $A$ be a Banach algebra. An element $a \in A$ is called left weakly completely continuous (l.w.c.c.) if the mapping $L_a$ defined by $L_a(x) = ax(X \in A)$ is weakly completely continuous. We say that $A$ is l.w.c.c. if each $a \in A$ is l.w.c.c. If $A$ is both l.w.c.c. and r.w.c.c., then $A$ is called w.c.c.

In this paper, all algebras and linear spaces under consideration are over the field $C$ of complex numbers.

3. THE MAIN RESULT.

LEMMA 3.1. Let $A$ be a Banach algebra. Then $A$ is l.w.c.c. (resp. r.w.c.c.) if and only if $\pi(A)$ is a right (resp. left) ideal of $(A^{**}, o)$.

PROOF. This result is well known (see [1, p.318, Lemma 3] or [2, p.443, Lemma 3]).

In the rest of this section, we shall assume that $A$ and $(A^{**}, o)$ are semisimple Banach algebras.

THEOREM 3.3. Suppose that $(A^{**}, o)$ has a dense socle. Then the following statements are equivalent:

1. $A$ is reflexive.
2. $A^{**}$ is w.c.c.
3. $A$ is w.c.c.
4. $\pi(A)$ and $A^{**}$ have the same socle.

PROOF.

(1) $\Rightarrow$ (2). Assume that $A$ is reflexive. Then $A^{(4)} = A^{**} = A$; in particular, $\pi(A)^{**}$ is a two-sided ideal of $A^{(4)}$. Hence by Lemma 3.1, $A^{**}$ is w.c.c.

(2) $\Rightarrow$ (3). Assume that $A^{**}$ is w.c.c. Then $\pi(A^{**})$ is a two-sided ideal of $A^{(4)}$. As observed in [1, p.319, Theorem 6(ii)], $\pi(A)$ is a two-sided ideal of $A^{**}$. Hence $A$ is w.c.c.

(3) $\Rightarrow$ (4). Assume that $A$ is w.c.c. Then $\pi(A)$ is a two-sided ideal of $A^{**}$. Let $E$ be a minimal idempotent of $A^{**}$. Since $E o A^{**} o E = E o \pi(A) o E = C E$, it follows that $E \in \pi(A)$. Consequently, $E$ is a minimal idempotent of $\pi(A)$. If $e$ is a minimal idempotent of $A$, then $\pi(e) o A^{**} \subset \pi(A)$ and so $\pi(e) o A^{**} = \pi(e A)$. Hence, $\pi(e) o A^{**} o \pi(e) = \pi(e A e) = C \pi(e)$ and so $e$ is a minimal idempotent of $A^{**}$. Therefore, $\pi(A)$ and $A^{**}$ have the same socle.

(4) $\Rightarrow$ (1). Assume that $\pi(A)$ and $A^{**}$ have the same socle. Since $\pi(S)$ is dense in $A^{**}$, it follows that $\pi(A)$ is dense in $A^{**}$ and so $\pi(A) = A^{**}$. Therefore $A$ is reflexive. This completes the proof of the theorem.

REMARK. It is well known that a semisimple annihilator Banach algebra $A$ is w.c.c. (see [5]). Also, $A$ has a dense socle. Therefore, Theorem 3.2 generalizes [1, p.319, Theorem 6(ii)].

THEOREM 3.3. If $A$ is 1.w.c.c., then $A$ is Arens regular.

PROOF. Since $A$ is 1.w.c.c., by Lemma 3.1, $\pi(A)$ is a right ideal of $A^{**}$. Let $F$ and $G \in A^{**}$ and $x \in A$. Then

$$\pi(x) o (F o G - F o' G) = \pi(x) o F o G - \pi(x) o (F o' G)$$

$$= \pi(x) o F o G - \pi(x) o (F o' G) \quad \text{By Lemma 2.1(1)}$$
\[ \pi(x)G - (\pi(x)F)G = 0 \]

Hence \(\pi(A)G(FG - F'G) = 0\). Therefore, by Lemma 2.1 (2), we have \(A^{**}G(FG - F'G) = 0\). Since \((A^{**}, o)\) is semisimple, it follows that \(FG - F'G = 0\) and so \(FG = F'G\). Therefore, A is Arens regular. This completes the proof.

REFERENCES


