REMARKS ON SEMISEPARATION OF LATTICES

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ABSTRACT. This paper is concerned primarily with conditions for semiseparation and separation of lattices. These conditions are expressed in terms of the general Wallman space.

KEY WORDS AND PHRASES. Lattice, measure, filter.

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1. INTRODUCTION.

Let X be an abstract set and \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) lattices of subsets of X such that \( \mathcal{L}_1 \subseteq \mathcal{L}_2 \). If \( A \cap B = \emptyset \), \( A \in \mathcal{L}_1 \), \( B \in \mathcal{L}_2 \) implies there exists a \( C \in \mathcal{L}_2 \), such that \( C \supseteq B \), and \( A \cap C = \emptyset \) then \( \mathcal{L}_1 \) is said to semiseparate \( \mathcal{L}_2 \). This notion is important in topological spaces, where \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are specific lattices such as, for example, the zero-sets and the closed sets.

We investigate this property in terms of associated measures and outer measures associated with the respective lattices, and also with respective Wallman spaces. This gives us new conditions for one lattice to semiseparate another, and gives additional facts pertaining to the measures. These investigations are carried out in sections 3 and 4. In section 2 we give some background material, which is fairly standard by now, and can be found in [1-3]. This material has been added mainly for the reader's convenience.

2. BACKGROUND AND NOTATIONS.

Let X be an abstract set and \( \mathcal{L} \) a lattice of subsets of X. It is assumed that \( \emptyset, X \in \mathcal{L} \). We denote by \( \mathcal{A}(\mathcal{L}) \) the algebra generated by \( \mathcal{L} \); \( \mathcal{D}(\mathcal{L}) \), the lattice of all countable intersections of sets from \( \mathcal{L} \).

DEFINITION 2.1. \( \mathcal{L} \) is:

delta lattice (\( \mathcal{D} \)-lattice) if \( \mathcal{L} \) is closed under countable intersections.

disjunctive generated if \( L \in \mathcal{L} \) implies \( L = \bigcap_{n=1}^{\infty} L_n \), \( L_n \in \mathcal{L} \) (where prime denotes complement).

disjunctive if for \( x \in X \) and \( L_1 \in \mathcal{L} \) such that \( x \notin L_1 \) there exists \( L_2 \in \mathcal{L} \) with \( x \in L_2 \) and \( L_1 \cap L_2 = \emptyset \).

normal if for any \( L_1, L_2 \in \mathcal{L} \), with \( L_1 \cap L_2 = \emptyset \), there exist \( L_3, L_4 \in \mathcal{L} \) with \( L_1 \subseteq L_3 \), \( L_2 \subseteq L_4 \) and \( L_1 \cap L_2 = L_3 \cap L_4 = \emptyset \).

compact if for any collection \( \{ L_\alpha \} \) of sets of \( \mathcal{L} \) with \( \bigcap_{\alpha} L_\alpha = \emptyset \), there exists a finite subcollection with empty intersection.
countably compact if for any countable collection \( \{L_n\} \) of sets of \( \mathcal{L} \) with \( \bigcap L_n = \emptyset \), there exists a finite subcollection with empty intersection.

Lindelöf if for any collection \( \{L_n\} \) of sets of \( \mathcal{L} \) with \( \bigcap L_n = \emptyset \), there exists a countable subcollection with empty intersection.

T$_2$-lattice if for \( x, y \in X \), \( x \neq y \), there exist \( L_1, L_2 \in \mathcal{L} \) such that \( x \in L_1 \), \( y \in L_2 \) and \( L_1 \cap L_2 = \emptyset \).

**DEFINITION 2.2** We give now some measure terminology which will be used throughout. \( M(\mathcal{L}) \) denotes the set of finite valued bounded finitely additive non-trivial measures on \( \mathcal{A}(\mathcal{L}) \). Without loss of generality may assume throughout that all measures are non-negative. A measure \( \mu \in M(\mathcal{L}) \) is called:

- \( \mathcal{G} \)-smooth on \( \mathcal{L} \) if for all sequences \( \{L_n\} \) of sets of \( \mathcal{L} \) with \( L_n \downarrow \emptyset \), \( \mu(L_n) \rightarrow 0 \).

- \( \mathcal{G} \)-smooth on \( \mathcal{A}(\mathcal{L}) \) if for all sequences \( \{A_n\} \) of sets of \( \mathcal{A}(\mathcal{L}) \) with \( A_n \downarrow \emptyset \), \( \mu(A_n) \rightarrow 0 \) (i.e. countably additive measures on \( \mathcal{A}(\mathcal{L}) \)).

- \( \mathcal{L} \)-regular if for any \( A \in \mathcal{A}(\mathcal{L}) \), \( \mu(A) = \sup \{\mu(L) : L \subseteq A, L \in \mathcal{L}\} \).

In addition we denote by \( M_{R}(\mathcal{L}) \), the set of \( \mathcal{L} \)-regular measures of \( M(\mathcal{L}) \); \( M_{\mathcal{G}}(\mathcal{L}) \) the set of \( \mathcal{G} \)-smooth measures on \( \mathcal{L} \) of \( M(\mathcal{L}) \); \( M_{R}^{\mathcal{G}}(\mathcal{L}) \), the set of \( \mathcal{G} \)-smooth measures on \( \mathcal{A}(\mathcal{L}) \) of \( M(\mathcal{L}) \).

\( I(\mathcal{L}), I_{R}(\mathcal{L}), I_{G}(\mathcal{L}), I_{R}^{G}(\mathcal{L}) \) are the subsets of the corresponding \( M \)'s which consist of the non-trivial zero-one valued measures.

**DEFINITION 2.3** For \( \mu \in M(\mathcal{L}) \), the support of \( \mu \) is \( S(\mu) = \{L \in \mathcal{L} : \mu(L) = \mu(X)\} \). \( \mathcal{L} \) is replete iff for any \( \mu \in I_{R}^{G}(\mathcal{L}) \), \( S(\mu) \neq \emptyset \).

**DEFINITION 2.4** A filter in \( \mathcal{L} \) is a subset of \( \mathcal{L} \), \( \mathcal{F} \), satisfying the conditions: \( \emptyset \notin \mathcal{F} \); \( \mathcal{F} \) is closed under finite intersections; if \( A \in \mathcal{F} \), \( B \in \mathcal{L} \) and \( A \subseteq B \) then \( B \in \mathcal{F} \).

An ultrafilter in \( \mathcal{L} \) is a maximal filter in \( \mathcal{L} \) (relative to the partial order on the collection of filters in \( \mathcal{L} \) given by inclusion).

An \( \mathcal{L} \)-filter \( \mathcal{F} \) is prime if given \( A, B \in \mathcal{L} \) such that \( A \cup B \in \mathcal{F} \) then either \( A \in \mathcal{F} \) or \( B \in \mathcal{F} \).

There exists a one-to-one correspondence between \( \mathcal{L} \)-filters \( \mathcal{F} \) and elements of \( \Pi(\mathcal{L}) = \{\Pi \}, \) defined on \( \mathcal{L} \), monotone and \( \Pi(A \cap B) = \Pi(A) \Pi(B), A, B \in \mathcal{L} \} \) defined by \( \Pi(L) = 1 \) iff \( L \in \mathcal{F} \). There exists a one-to-one correspondence between \( \mathcal{L} \)-filters \( \mathcal{F} \) with countable intersection property and \( \Pi_{R}(\mathcal{L}) \), where \( \Pi_{R}(\mathcal{L}) = \{\Pi \in \Pi(\mathcal{L}) \) such that if \( \Pi(L_n) = 1 \) all \( n \) where \( L_n \in \mathcal{L} \) then \( \bigcap L_n \neq \emptyset \}. \) There exists a one-to-one correspondence between all elements of \( I_{R}(\mathcal{L}) \) and all \( \mathcal{L} \)-ultrafilters. There exists a one-to-one correspondence between all elements of \( I_{R}^{G}(\mathcal{L}) \) and all \( \mathcal{L} \)-ultrafilters with the countable intersection property. The correspondence is given by the following rule: with each \( \mathcal{L} \)-ultrafilter \( \mathcal{F} \) we associate the zero-one valued measure defined on \( \mathcal{A}(\mathcal{L}) \) by

\[
\mu(E) = \begin{cases} 
1 & \text{if there exists } A \in \mathcal{F}, A \subseteq E \\
0 & \text{if there exists } A \in \mathcal{F}, A \subseteq \mathcal{L} \setminus E
\end{cases}
\]

There exists a one-to-one correspondence between all elements of \( I(\mathcal{L}) \) and all prime \( \mathcal{L} \)-filters, given by the following rule: with each \( \mu \in I(\mathcal{L}) \) we associate the prime \( \mathcal{L} \)-filter given by \( \mathcal{F} = \{A \in \mathcal{L} : \mu(A) = 1\} \). This correspondence induces a one-to-one correspondence between prime \( \mathcal{L} \)-filters with the countable intersection property and \( I_{G}(\mathcal{L}) \).

**REMARK.** It is not difficult to see in light of the above correspondences that \( \mathcal{L} \) is normal iff for each \( \mu \in I(\mathcal{L}) \), there exists a unique \( \nu \in I_{R}(\mathcal{L}) \) such that \( \mu = \nu \) (i.e. \( \mu(L) \leq \nu(L) \) for all \( L \in \mathcal{L} \)).
3. SEMISEPARATION.

DEFINITION 3.1 Let \( \mathcal{L} \) be a lattice of subsets of \( X \), let \( \mu \in I(\mathcal{L}) \) and \( E \subseteq X \) and define \( \mu'(E) = \inf \{ \mu(L')/ E \subseteq L' \} \).

THEOREM 3.1 Let \( \mathcal{L} \) be a lattice of subsets of \( X \) and let \( \mu \in I(\mathcal{L}) \). The following statements are true:

a) \( \mu' \) is finitely subadditive;

b) \( \mu \in I'_R(\mathcal{L}) \) iff \( \mu = \mu'(\mathcal{L}) \);

c) Let \( \mu \in I'_R(\mathcal{L}) \) and \( \rho \in I'_R(\mathcal{L}) \) such that \( \mu \leq \rho \). \( \mathcal{L} \) is normal iff \( \mu' = \rho'(\mathcal{L}) \) for all such \( \mu \) and \( \rho \).

PROOF. a) Since \( \mu \in I(\mathcal{L}) \) it is clear that \( \mu'(E) = \inf \{ \mu(L')/ E \subseteq L' \} \) and therefore \( \mu' \) is finitely subadditive.

b) For \( \mu \in I'_R(\mathcal{L}) \), \( \mu(A) = \inf \{ \mu(L')/ A \subseteq L' \} \). \( \mu(A) \in I(\mathcal{L}) \) and since \( \mu \in I'_R(\mathcal{L}) \) it follows that \( \mu = \mu'(\mathcal{L}) \). Then \( \mu \leq \rho \) on \( \mathcal{L} \). Suppose \( \mathcal{L} \) is normal, let \( A \subseteq \mathcal{L} \) and suppose that \( \mu(A) = 0 \). Since \( \mu \in I'_R(\mathcal{L}) \), there exists \( L \subseteq A' \), \( L \subseteq \mathcal{L} \) with \( \mu(L) = 1 \). But \( A \subseteq \mathcal{L} \) implies there exist \( C', D \subseteq \mathcal{L} \) such that \( A \subseteq C', \mathcal{L} \subseteq D' \) and \( C(D') \subseteq D = \mu(L) = 0 \). So, \( \mu(C') = 0 \), i.e. \( \mu'(A) = 0 \). A was arbitrary in \( \mathcal{L} \), then \( \mu' = \rho' \) on \( \mathcal{L} \). Conversely, suppose that \( \mu = \rho' \) on \( \mathcal{L} \) with \( \mu \) and \( \rho \) as before. Let \( \mu \in I(\mathcal{L}) \), \( \mu \leq \rho \) on \( \mathcal{L} \), so we have \( \rho \leq \rho' \) on \( \mathcal{L} \) and \( \rho \leq \rho \) on \( \mathcal{L} \). By the assumption, \( \mu = \rho' \) on \( \mathcal{L} \) and therefore \( \mu = \rho \).

definition 3.2 Let \( \mathcal{L} \) be a lattice of subsets of \( X \). The Wallman topology is obtained by taking all \( W(L) = \{ \mu \in I'_R(\mathcal{L})/ \mu(L) = 1 \} \), \( \mu \in \mathcal{L} \) as a base for the closed sets in \( I'_R(\mathcal{L}) \). \( I'_R(\mathcal{L}) \) with the Wallman topology is called the general Wallman space associated with \( X \) and \( \mathcal{L} \). We assume that \( \mathcal{L} \) is disjunctive. Then if \( A \subseteq \mathcal{L} \), let \( W(A) = \{ \mu \in I'_R(\mathcal{L})/ \mu(A) = 1 \} \). The following statements are true:

a) \( W(A) \cup W(B) = W(A \cup B) \)

b) \( W(A \cap B) = W(A) \cap W(B) \)

c) \( W(A') = W(A)' \)

d) \( A \supseteq B \) iff \( W(A) \supseteq W(B) \)

e) \( \mathcal{A}(W(\mathcal{L})) = W(\mathcal{A}(\mathcal{L})) \).

It is known that \( W(\mathcal{L}) \) is disjunctive and that the topological space \( (I'_R(\mathcal{L}), W(\mathcal{L})) \) is compact and \( T_1 \) and if \( \mathcal{L} \) is disjunctive it is \( T_2 \) iff \( \mathcal{L} \) is normal.

THEOREM 3.2 Let \( \mathcal{L}_1 \subseteq \mathcal{L}_2 \) be two lattices of subsets of \( X \). Suppose that \( \mathcal{L}_2 \) is disjunctive and \( \mathcal{L}_1 \) is normal and consider the restriction map \( \Psi : I'_R(\mathcal{L}_2) \rightarrow I'_R(\mathcal{L}_1) \). Then:

a) \( \Psi(W_2(L_2)) = W_1(L_1) \) where \( W_1(L_1) \) and \( W_2(L_2) \) are basic closed sets with respect to the Wallman topologies.

b) \( \mathcal{L}_2 \) semiseparates \( \mathcal{L}_1 \).

PROOF. a) Since \( W_2(L_2) \) is closed in \( W_2(\mathcal{L}_2) \), it is compact and since \( \Psi \) is continuous, \( \Psi(W_2(L_2)) \) is compact. \( \Psi(W_2(L_2)) \) is normal, so \( I'_R(\mathcal{L}_2) \) is compact and \( T_2 \) and therefore \( \Psi(W_2(L_2)) \) is closed. Then \( \Psi(W_2(L_2)) = \bigcup W_1(L_1) \), where \( L_1 \subseteq \mathcal{L}_1 \) and \( L_2 \subseteq \mathcal{L}_2 \) are basic closed sets with respect to the Wallman topologies.

b) \( \mathcal{L}_2 \) semiseparates \( \mathcal{L}_1 \).

PROOF. a) Since \( W_2(L_2) \) is closed in \( W_2(\mathcal{L}_2) \), it is compact and since \( \Psi \) is continuous, \( \Psi(W_2(L_2)) \) is compact. \( \Psi(W_2(L_2)) \) is normal, so \( I'_R(\mathcal{L}_2) \) is compact and \( T_2 \) and therefore \( \Psi(W_2(L_2)) \) is closed. Then \( \Psi(W_2(L_2)) = \bigcup W_1(L_1) \), where \( L_1 \subseteq \mathcal{L}_1 \) and \( L_2 \subseteq \mathcal{L}_2 \) are basic closed sets with respect to the Wallman topologies.

b) Suppose \( L_1 \subseteq \mathcal{L}_1 \) and \( L_2 \subseteq \mathcal{L}_2 \) with \( L_1 \subseteq L_2 \). Since \( L_1 \subseteq \mathcal{L}_1 \) and \( L_2 \subseteq \mathcal{L}_2 \), it follows that \( \Psi(W_2(L_2)) \) is compact.

It is known that \( \Psi \) is continuous and that the topological space \( (I'_R(\mathcal{L}_2), W(\mathcal{L}_2)) \) is compact and \( T_1 \) and if \( \mathcal{L}_2 \) is disjunctive it is \( T_2 \) iff \( \mathcal{L}_2 \) is normal.

THEOREM 3.2 Let \( \mathcal{L}_1 \subseteq \mathcal{L}_2 \) be two lattices of subsets of \( X \). Suppose that \( \mathcal{L}_2 \) is disjunctive and \( \mathcal{L}_1 \) is normal and consider the restriction map \( \Psi : I'_R(\mathcal{L}_2) \rightarrow I'_R(\mathcal{L}_1) \). Then:

a) \( \Psi(W_2(L_2)) = W_1(L_1) \) where \( W_1(L_1) \) and \( W_2(L_2) \) are basic closed sets with respect to the Wallman topologies.

b) \( \mathcal{L}_2 \) semiseparates \( \mathcal{L}_1 \).

PROOF. a) Since \( W_2(L_2) \) is closed in \( W_2(\mathcal{L}_2) \), it is compact and since \( \Psi \) is continuous, \( \Psi(W_2(L_2)) \) is compact. \( \Psi(W_2(L_2)) \) is normal, so \( I'_R(\mathcal{L}_2) \) is compact and \( T_2 \) and therefore \( \Psi(W_2(L_2)) \) is closed. Then \( \Psi(W_2(L_2)) = \bigcup W_1(L_1) \), where \( L_1 \subseteq \mathcal{L}_1 \) and \( L_2 \subseteq \mathcal{L}_2 \) are basic closed sets with respect to the Wallman topologies.

b) \( \mathcal{L}_2 \) semiseparates \( \mathcal{L}_1 \).
that \( \bigcap_{e \in L_1} W_1(L_1) = L_2 \). Then \( L_2 \subseteq \bigcap_{e \in L_1} W_1(L_1) = A \in \mathcal{G} \), and \( A \cap L_1 = \emptyset \) which proves that \( \mathcal{G} \) separates \( \mathcal{L}_2 \).

**COROLLARY 3.1** Let \( \mathcal{L} \) be a lattice of subsets of \( X \). Then the following statements are equivalent:

- a) \( \mathcal{I}(\mathcal{L}) = \mathcal{I}(\mathcal{L}) \)
- b) \( \mathcal{L} \) semisepares \( \mathcal{A}(\mathcal{L}) \)
- c) \( \mathcal{L} = \mathcal{L}' \)
- d) \( \mathcal{I}(\mathcal{L}') = \mathcal{I}(\mathcal{L}) \)

**PROOF.** a) \( \Rightarrow \) b): \( \mathcal{A}(\mathcal{L}) \) is disjunctive. \( \mathcal{L} \) is normal, since for \( \mu \in \mathcal{I}(\mathcal{L}) = \mathcal{I}(\mathcal{L}) \) we have \( \mu \leq \mu(\mathcal{L}) \). Consider the restriction map \( \Psi : \mathcal{I}(\mathcal{A}(\mathcal{L})) = \mathcal{I}(\mathcal{A}(\mathcal{L})) \rightarrow \mathcal{I}(\mathcal{A}(\mathcal{L})) \). By Theorem 3.2 it follows that \( \mathcal{L} \) semisepares \( \mathcal{A}(\mathcal{L}) \).

- b) \( \Rightarrow \) c): Let \( L \in \mathcal{L} \), then \( L' \in \mathcal{A}(\mathcal{L}) \). Since \( \mathcal{L} \) semisepares \( \mathcal{A}(\mathcal{L}) \) there exists \( A \in \mathcal{L} \), \( L' \subseteq A \) and \( A \cap L_1 = \emptyset \), i.e. \( A \subseteq L' \). Therefore \( L \subseteq A \subseteq L' \), i.e. \( \mathcal{L} = \mathcal{L}' \).

- c) \( \Rightarrow \) d), clearly. d) \( \Rightarrow \) a): Let \( \mu \in \mathcal{I}(\mathcal{L}) \) and \( \Psi \in \mathcal{I}(\mathcal{L}), \mu \leq \Psi(\mathcal{L}) \) and suppose that \( \mu(\mathcal{L}) = 0, \Psi(\mathcal{L}) = 1 \). But \( \Psi \in \mathcal{I}(\mathcal{L}'), \) therefore there exists \( L \subseteq A \), \( L' \subseteq L_1 \) with \( \Psi(L') = 1 \), or \( \Psi(L) = 0 \). Then \( \mu(L) = 0 \) and since \( \Psi(\mathcal{L}'), \mu(A) \leq \mu(L) = 0 \) i.e. \( \mu(A') = 0 \), contradiction. It follows that \( \mu = \Psi \) i.e. \( \mathcal{I}(\mathcal{L}) = \mathcal{I}(\mathcal{L}) \).

**COROLLARY 3.2** Let \( \mathcal{L}_1 \subseteq \mathcal{L}_2 \) be two lattices of subsets of \( X \), with \( \mathcal{L}_1 \) normal and \( \mathcal{L}_2 \) disjunctive. Consider \( \Psi \in \mathcal{I}(\mathcal{L}_2) \) and its restriction \( \Psi(\mathcal{L}_1) \). Then \( \Psi \rightarrow \mu(\mathcal{L}_1) \) iff \( \mathcal{L}_1 \) semisepares \( \mathcal{L}_2 \).

**PROOF.** Clearly, \( \Psi \rightarrow \mu(\mathcal{L}_1) \), always. Let \( L_1 \subseteq \mathcal{L}_1 \) and suppose \( \Psi(L_1) = 0 \). Then \( L_1 \subseteq L_2 \) and \( \Psi(L_1) = 0 \) by semisepaion, there exists \( L_1 \subseteq \mathcal{L}_1 \), \( L_2 \subseteq \mathcal{L}_2 \) and \( \Psi(L_1) = 0 \). Then \( L_1 \subseteq L_2 \) and \( \Psi(L_1) = 0 \), i.e. \( \mu(L_1) = 0 \) and \( \Psi(L_1) = 0 \). Conversely, suppose \( \Psi \rightarrow \mu(\mathcal{L}_1) \). If \( \mu(L_1) = 0 \) then \( \Psi(L_1) = 0 \) therefore \( \Psi(L_1) = 0 \), \( L_1 \subseteq \mathcal{L}_1 \), since \( \Psi = \Psi(\mathcal{L}_1) \). So \( \mu(L_1) = 0 \) i.e. \( \mu = \mu(\mathcal{L}_1) \) which by Theorem 3.1 implies \( \mu \in \mathcal{I}(\mathcal{L}_2) \).

It follows by Theorem 3.2 that \( \mathcal{L}_1 \) semisepares \( \mathcal{L}_2 \).

**DEFINITION 3.3** Let \( \mathcal{L} \) be a lattice of subsets of \( X \) and define

\[ \mathcal{E}(\mathcal{L}) = \inf \{ \mu(L), \mu \in \mathcal{L}, L \subseteq X \}. \]

**THEOREM 3.3** Let \( \mathcal{L}_1 \subseteq \mathcal{L}_2 \) be two lattices of subsets of \( X \) and let \( \mu \in \mathcal{I}(\mathcal{L}_2) \). Then \( \mu = \mu(\mathcal{L}_2) \) iff \( \mathcal{L}_1 \) semisepares \( \mathcal{L}_2 \).

**PROOF.** \( \mu(\mathcal{L}_2) = \inf \{ \mu(L_1), L_1 \subseteq \mathcal{L}_1, L_1 \subseteq L_2 \}. \) By semisepaion, there exists \( L_1 \subseteq \mathcal{L}_1 \), \( L_2 \subseteq \mathcal{L}_2 \). Now suppose \( \mu(\mathcal{L}_2) = 0 \). Then there exists \( A \subseteq \mathcal{L}_1 \), \( L_2 \subseteq A \) and \( \mu(A) = 0 \) and since \( \mu \in \mathcal{I}(\mathcal{L}_2) \), there exists \( B \subseteq \mathcal{L}_1 \), \( A \subseteq B \) and \( \mu(B') = 0 \). Therefore \( L_2 \subseteq B \) and \( \mu(B') = 0 \), hence \( \mu(\mathcal{L}_2) = 0 \). So \( \mu = \mu(\mathcal{L}_2) \).

Conversely, suppose that \( \mu = \mu(\mathcal{L}_2) \) for all \( \mu \in \mathcal{I}(\mathcal{L}_1) \). If \( \mathcal{L}_1 \) does not semisepares \( \mathcal{L}_2 \) then there exists \( L_1 \subseteq \mathcal{L}_1 \), \( L_1 \subseteq \mathcal{L}_2 \) such that \( \mu(L_1) = 0 \) but \( \mu(L_2) = 0 \). But \( \mu(L_2) = 0 \), contradiction. Hence \( \mathcal{L}_1 \) semisepares \( \mathcal{L}_2 \).

**DEFINITION 3.4** Let \( \mathcal{L} \) be a lattice of subsets of \( X \), let \( \mu \in \mathcal{I}(\mathcal{L}) \) and \( E \subseteq X \) and define

\[ \mathcal{E}(E) = \inf \{ \mu(L), E \subseteq \mathcal{L}, L \subseteq E \}. \]

**DEFINITION 3.5** Let \( \mathcal{L} \) be a lattice of subsets of \( X \), let \( \mu \in \mathcal{I}(\mathcal{L}) \) and \( E \subseteq X \) and define

\[ \mathcal{E}(E) = \inf \{ \mu(L), E \subseteq \mathcal{L}, L \subseteq E \}. \]

**REMARK** Both \( \mu \) and \( \mathcal{E} \) are outer measures on \( P(X) \), clearly. If \( \mu \in \mathcal{I}(\mathcal{L}) \) then \( \mu = 0 \). If \( \mu \in \mathcal{I}(\mathcal{L}) \) then \( \mu = \mu(\mathcal{L}) \). Similar remarks for \( \mathcal{E} \).

**THEOREM 3.4** Let \( \mathcal{L}_1 \subseteq \mathcal{L}_2 \) be two lattices of subsets of \( X \) such that \( \mathcal{L}_1 \) semisepares \( \mathcal{L}_2 \). Suppose that \( \mathcal{L}_1 \) is \( \mathcal{F} \) and let \( \mu \in \mathcal{I}(\mathcal{L}_1) \). Then \( \mu = \mu(\mathcal{L}_2) \).
PROOF. Clearly, \( \mu = \mu'\). Let \( \mathcal{L} \) be a \( \mathcal{L} \)-lattice so that \( L = \bigcap L_i \), \( L \in \mathcal{L} \), and let \( \mu \in \mathcal{L} \). Then \( \mu(\bigcap L_i) = \mu((\bigcap L_i)) = \mu((\bigcap L_i)) \leq \mu(L_i) \) since \( \mu \) is countably additive; therefore \( \mu = \mu'\). By Theorem 3.3 it follows \( \mu = \mu'\). But since \( \mu \in \mathcal{L} \) everywhere, we get \( \mu = \mu'\). Suppose \( \mu(L_2) = 0 \) with \( L_2 \in \mathcal{L} \), but \( \mu(L_2) = 1\). Then \( L_2 \subseteq A_1 \subseteq B_i \), \( A_i \in \mathcal{L} \), and \( \mu(A_i) = 0 \). By the \( \mathcal{L} \)-regularity of \( \mu \) we have \( A_i \subseteq B_i \), \( \mu(B_i) = 0 \), and \( \mu''(B_i) = 0 \). Therefore \( L_2 \subseteq B_i \), \( \mu(B_i) = 0 \) and \( \mu''(B_i) = 0 \), contradiction. Hence \( \mu = \mu'(L_2)\).

Further related material can be found in [4-6].

4. \( \mathcal{L} \)-LATTICES

DEFINITION 4.1 A lattice \( \mathcal{L} \) is called an \( \mathcal{L} \)-lattice if every \( \mathcal{L} \)-filter with the countable intersection property is contained in an \( \mathcal{L} \)-ultrafilter with the countable intersection property (i.e. for \( \mathcal{L} \in \mathcal{L} \), there exists \( \mu \in \mathcal{L} \) such that \( \mu(\mathcal{L}) \) and \( \mu' = \mu(\mathcal{L}) \)).

THEOREM 4.1 If \( \mathcal{L} \) is an \( \mathcal{L} \)-lattice and \( \mathcal{L} \) is Lindelöf.

PROOF. Let \( \mu \in \mathcal{L} \). There exists \( \mu \in \mathcal{L} \) with \( \mu = \mu(\mathcal{L}) \). \( \mathcal{L} \) replete implies \( \mu(\mathcal{L}) = 1 \) and \( \mu(\mathcal{L}) = 1 \). Therefore \( \mu = \mu(\mathcal{L}) \).

THEOREM 4.2 If \( \mathcal{L} \) is a countably compact lattice then \( \mathcal{L} \) is an \( \mathcal{L} \)-lattice.

PROOF. Let \( \mathcal{L} \) be a countably compact lattice then \( \mathcal{L} \) is an \( \mathcal{L} \)-lattice.

DEFINITION 4.2 Let \( \mathcal{L} \) be a disjunctive lattice of subsets of \( X \) and let \( \mu \in \mathcal{L} \). Define \( \mu' \) on \( \mathcal{L}(\mathcal{L}) = \lambda(\mathcal{L}(\mathcal{L})) \) by \( \mu' = \mu(\mathcal{L}) \), \( \mu(A) = \mu(\mathcal{L}) \), \( A \subseteq \mathcal{L}(\mathcal{L}) \) where \( \mathcal{L}(\mathcal{L}) = \lambda(\mathcal{L}) \). Clearly, for \( A, B \in \mathcal{L}(\mathcal{L}) \) the properties a-e) that we stated in section 3 are still valid. Note that \( \mathcal{L}(\mathcal{L}) \) is a disjunctive lattice.

The following theorem follows directly from the definitions:

THEOREM 4.4 If \( \mu \in \mathcal{L} \) then \( \mu = \mu'(\mathcal{L}) \). (More generally: if \( \mu \in \mathcal{L} \) then \( \mu = \mu'(\mathcal{L}) \).)

THEOREM 4.5 If \( \mathcal{L} \) is disjunctive then \( \mathcal{L} \) is an \( \mathcal{L} \)-lattice iff \( \mathcal{L}(\mathcal{L}) \) is Lindelöf.

PROOF. Necessity: first we show that \( \mathcal{L}(\mathcal{L}) \) is an \( \mathcal{L} \)-lattice. Let \( \mu \in \mathcal{L}(\mathcal{L}) \).

There exists \( \mu = \mu(\mathcal{L}) \) with \( \mu(\mathcal{L}) = \mu(\mathcal{L}) \). Hence by Theorem 4.4 we have \( \mu = \mu(\mathcal{L}) \) and \( \mu = \mu(\mathcal{L}) \). Since \( \mathcal{L} \) is disjunctive, \( \mathcal{L}(\mathcal{L}) \) is replete and by Theorem 4.1 it follows that \( \mathcal{L}(\mathcal{L}) \) is Lindelöf. \( \mathcal{L}(\mathcal{L}) \subseteq \mathcal{L}(\mathcal{L}) \) implies that \( \mathcal{L}(\mathcal{L}) \) is Lindelöf. Sufficiency: \( \mathcal{L}(\mathcal{L}) \) Lindelöf implies that \( \mathcal{L}(\mathcal{L}) \) Lindelöf and since \( \mathcal{L}(\mathcal{L}) \) is disjunctive, by Theorem 4.3 it follows that \( \mathcal{L}(\mathcal{L}) \) is an \( \mathcal{L} \)-lattice.

Therefore for \( \mathcal{L}(\mathcal{L}) \) there exists \( \mu = \mu(\mathcal{L}) \) such that \( \mu = \mu(\mathcal{L}) \). To \( \mu \) and \( \mu' \) correspond \( \mu = \mu(\mathcal{L}) \) and \( \mu = \mu(\mathcal{L}) \) such that \( \mu = \mu(\mathcal{L}) \).

THEOREM 4.6 Let \( \mathcal{L}_1 \subseteq \mathcal{L}_2 \) be two lattices of subsets of \( X \) such that \( \mathcal{L}_1 \) is an \( \mathcal{L} \)-lattice and \( \mathcal{L}_2 \) is disjunctive. Consider that the restriction \( \mathcal{L}_1 \) is closed with respect to Wallman topologies. Then \( \mathcal{L}_1 \) semiseparates \( \mathcal{L}_2 \).
PROOF. Let $L_1 \in \mathcal{L}_1$, $L_2 \in \mathcal{L}_2$ with $L_1 \cap L_2 = \emptyset$. Then in $\mathcal{G}_R(L_2)$: $\mathcal{G}_R(L_2) \cap \mathcal{G}_R(L_1) = \emptyset$ and in $\mathcal{G}_R(L_1)$: $\mathcal{G}_R(L_1) \cap \mathcal{G}_R(L_1') = \emptyset$ for $\mu \in \mathcal{G}_R(L_2)$ and if $\mu \in \mathcal{G}_R(L_1)$ then $\mu = \mu(\mathcal{G}_R(L_2))$ where $\mathcal{G}_R(L_2) = \emptyset$ and $\mu(L_1) = 1$. But then $\mathcal{G}_R(L_2) = 1$ and $\mathcal{G}_R(L_1) = 1$, contradiction since $L_1 \cap L_2 = \emptyset$. Now since $\mathcal{G}_R(L_2)$ is closed $\mathcal{G}_R(L_2) = \mathcal{G}_R(L_1)$, $L_2 \subseteq L_1$ $\subseteq L_1'$ therefore $\bigcap_{\mu \in \mathcal{G}_R(L_1')} \mathcal{G}_R(L_1) = \emptyset$. Hence, since $L_1'$ is an I-lattice and disjunctive (because $\mathcal{G}_R(L_2)$ is disjunctive), by Theorem 4.5 $\mathcal{G}_R(L_1)$ is Lindelöf. Now $\mathcal{G}_R(L_1') \cap \mathcal{G}_R(L_1) = \emptyset$ and then $\bigcap_{\mu \in \mathcal{G}_R(L_1')} \mu(L_1) = \emptyset$, where $\bigcap_{\mu \in \mathcal{G}_R(L_1')} \mu(L_1)$ which is $\mathcal{F}$ and $\bigcap_{\mu \in \mathcal{G}_R(L_1')} \mu(L_1)$ separates $\mathcal{L}_2$. Hence $L_1$ semiseparates $\mathcal{L}_2$.

Here we give conditions which guarantee that $\mathcal{G}_R(L_2)$ is basically closed.

DEFINITION 4.3 $\mathcal{L}_2$ is countably bounded $\mathcal{L}_1$ lattice if for $A_n \in \mathcal{L}_2$, $n = 1, 2, \ldots$ and $A_n \uparrow \emptyset$, there exists $B_n \subseteq L_1$, $n = 1, 2, \ldots$ with $A_n \subseteq B_n$ and $B_n \uparrow \emptyset$.

THEOREM 4.7 Let $\mathcal{L}_1 \subseteq \mathcal{L}_2$ be two lattices of subsets of $X$ such that $\mathcal{L}_1$ semiseparates $\mathcal{L}_2$, $\mathcal{L}_2$ is $\mathcal{L}_1$-countably bounded and $\mathcal{L}_2 = t \mathcal{L}_1$. Then the restriction $\mathcal{G}_R(L_2) = \bigcap_{\mu \in \mathcal{G}_R(L_1')} \mathcal{G}_R(L_1)$ is basically closed.

PROOF. To show that $\mathcal{G}_R(L_2) = \bigcap_{\mu \in \mathcal{G}_R(L_1')} \mathcal{G}_R(L_1)$ is countably bounded. Clearly $\mathcal{G}_R(L_2) = \bigcap_{\mu \in \mathcal{G}_R(L_1)} \mathcal{G}_R(L_1)$, $L_2 \subseteq L_1 \subseteq L_1'$, since $\mathcal{L}_2$ is countably bounded, $\mathcal{G}_R(L_2) = \bigcap_{\mu \in \mathcal{G}_R(L_1)} \mathcal{G}_R(L_1)$. Now let $\mu = \mathcal{G}_R(L_2)$, but $\mu = \mathcal{G}_R(L_1)$, therefore $\mu = \mathcal{G}_R(L_1)$ and since $\mathcal{G}_R(L_1)$ is countably bounded, $\mathcal{G}_R(L_1) = \mathcal{G}_R(L_1')$. So, $\mu(L_1) = 1$ and $\mu(L_1') = 1$. Since then $\mathcal{G}_R(L_2) = \emptyset$ and $\mu = \mathcal{G}_R(L_1) = \emptyset$. Therefore $\mu = \mathcal{G}_R(L_1)$, there exists $L_2 \subseteq L_1$, $L_2 \subseteq L_1'$, $\mu(L_2) = 1$. By semiseparation there exists $L_2 \subseteq L_1$, $L_2 \subseteq L_1'$, $\mu(L_2) = 1$. But $L_2 \subseteq L_1$ and $\mu(L_2) = 1$ implies $\mu(L_1) = 1$ and since also $\mu(L_1) = 1$ all $L_1$, it follows $\mu(L_1 \cap L_1') = 1$ which contradicts that $L_1 \cap L_1' = \emptyset$.

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