BEST APPROXIMATION IN ORLICZ SPACES

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(Received April 17, 1989 and in revised form December 23, 1989)

ABSTRACT. Let X be a real Banach space and (Ω,μ) be a finite measure space and ϕ be a strictly increasing convex continuous function on [0,∞) with ϕ(0) = 0. The space $L_ϕ(μ,X)$ is the set of all measurable functions $f$ with values in X such that

$$\int_Ω ϕ(\|c^{-1}f(t)\|)dμ(t) < \infty$$

for some $c > 0$. One of the main results of this paper is:

"For a closed subspace Y of X, $L_ϕ(μ,Y)$ is proximinal in $L_ϕ(μ,X)$ if and only if $L^1(μ,Y)$ is proximinal in $L^1(μ,X)"". As a result if Y is reflexive subspace of X, then $L_ϕ(μ,Y)$ is proximinal in $L_ϕ(μ,X)$. Other results on proximinality of subspaces of $L_ϕ(μ,X)$ are proved.

1. INTRODUCTION.

Let $ϕ$ be a convex Orlicz function, i.e., $ϕ$ is a continuous, strictly increasing convex function defined on [0,∞) with $ϕ(0) = 0$ and let $(Ω,μ)$ be a finite measure. For a real Banach space X, let

$L_ϕ(μ,X) = \{\text{measurable function } f: Ω \to X: \int_Ω ϕ(\|c^{-1}f(t)\|)dμ(t) < \infty \}$

for some $c > 0$. Define a norm on $L_ϕ(μ,X)$ by

$$\|f\|_ϕ = \inf \{c > 0: \int_Ω ϕ(\|c^{-1}f(t)\|)dμ(t) < 1\}.$$

A subspace Y in a Banach space X is called proximinal if for each $x \in X$ there is at least one $y \in Y$ such that $\|x - y\| = d(x,y) = \inf \{\|x - h\|: h \in Y\}$. The element $y$ is called best approximant of $x$ in $Y$. Set $P(x,Y) = \{y \in Y: d(x,y) = \|x - y\|\}$.

In this paper we prove that for a closed subspace Y of a Banach space X, $L_ϕ(μ,Y)$ is proximinal in $L_ϕ(μ,X)$ if and only if $L^1(μ,Y)$ is proximinal in $L^1(μ,X)$. In [1] Deeb and Khalil, have shown the same result for the linear metric space $L_ϕ(μ,X)$ with ϕ modulus function and some Banach space X. As a consequence, if Y

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is a reflexive subspace of a Banach space $X$ then $L_{\phi}(\mu, Y)$ is proximinal in $L_{\phi}(\mu, X)$.

The proximinality of some closed subspaces in $X$ are discussed. Throughout this paper $\Omega$ will be the unit interval $[0,1]$, $\phi$ convex, strictly increasing with $
abla(0) = 0$, $\phi(1) = 1$ and $X$ is a Banach space. See Deeb and Khalil [1,2,3], Light and Cheney [4], and Khalil [5] for more details about proximinality and related topics.

2. PROXIMALITY IN $L_{\phi}(\mu, X)$.

LEMMA 2.1. If $\phi$ is convex, then $L_{\phi}(\mu, X) \subseteq L^1(\mu, X)$.

PROOF. Let $f \in L_{\phi}(\mu, X)$, then

$$\int_0^1 \phi(||c^{-1}f(t)||)d\mu(t) < M$$

for some $c$ and some $M$.

By Jensen's Inequality, [6]

$$\phi\left(\int_0^1 ||c^{-1}f(t)||d\mu(t)\right) \leq \int_0^1 \phi(||c^{-1}f(t)||)d\mu(t) < M$$

or

$$\phi\left(\int_0^1 ||c^{-1}f(t)||d\mu(t)\right) < M.$$ 

Hence

$$\int_0^1 ||c^{-1}f(t)||d\mu(t) < \phi^{-1}(M).$$

Therefore

$$\int_0^1 ||f(t)||d\mu(t) < c^{-1}\phi^{-1}(M) < \infty.$$ 

Hence $f \in L^1(\mu, X)$.

LEMMA 2.2. Let $Y$ be a subspace of $X$, then for each $f \in L_{\phi}(\mu, X)$

$$\text{dist}(f, L_{\phi}(\mu, Y)) = \inf\{c > 0 : \int_0^1 \phi||c^{-1}\text{dist}(f(t), Y)||d\mu(t) < 1\}.$$ 

PROOF. For any $g \in L_{\phi}(\mu, Y)$ we have,

$$||f-g||_{\phi} = \inf\{c > 0 : \int_0^1 \phi(||c^{-1}(f(t) - g(t))||)d\mu(t) < 1\}$$

$$> \inf\{c > 0 : \int_0^1 \phi(||c^{-1}\text{dist}(f(t), Y)||)d\mu(t) < 1\}.$$ 

By taking the infimum over $g \in L_{\phi}(\mu, Y)$ we get

$$\text{dist}(f, L_{\phi}(\mu, Y)) > \inf\{c > 0 : \int_0^1 \phi(||c^{-1}\text{dist}(f(t), Y)||)d\mu(t) < 1\}. $$
Conversely, let $\epsilon > 0$ and let $f'$ be a simple function in $L_\Phi(u, X)$, such that

$$\|f-f'\|_\Phi < \epsilon.$$ 

Write $f' = \sum_{i=1}^{n} x_i x_i$, where $x_i \in X$ and $x_i$ are the characteristic functions on $A_i$ which are disjoint measurable sets in $[0,1]$. It is clear that $f' \in L_\Phi(u, X)$. Select $h_i \in Y$ such that

$$\phi(c^{-1}\|x_i - h_i\|) < \phi[c^{-1}\text{dist}(x_i, Y) + \epsilon], \quad \text{for some } c > 0.$$

Let $g = \sum_{i=1}^{n} x_i h_i$, then

$$\int_0^1 \phi(\|c^{-1}g(t)\|)d\mu(t) = \sum_{i=1}^{n} \phi(\|c^{-1}h_i\|)\mu(A_i) < \infty.$$ 

Hence $g \in L_\Phi(u, Y)$, then

$$\|f-g\|_\Phi = \|f-f' + f' - g\|_\Phi < \epsilon + \|f' - g\|_\Phi.$$ 

But $\text{dist}(f, L_\Phi(u, Y)) < \|f-g\|_\Phi$

$$< \epsilon + \inf \{c > 0 : \int_0^1 \phi(c^{-1}\|f'(t)-g(t)\|)d\mu(t) < 1\}$$ 

$$= \epsilon + \inf \{c > 0 : \sum_{i=1}^{n} \int_0^1 \phi(c^{-1}\|x_i - h_i\|)d\mu(t) < 1\}$$ 

$$= \epsilon + \inf \{c > 0 : \sum_{i=1}^{n} \phi(c^{-1}\|x_i - h_i\|)\mu(A_i) < 1\}$$ 

$$< \epsilon + \inf \{c > 0 : \sum_{i=1}^{n} \phi[c^{-1}\text{dist}(x_i, Y) + \epsilon]\mu(A_i) < 1\}$$ 

$$= \epsilon + \inf \{c > 0 : \int_0^1 \phi(c^{-1}\text{dist}(f'(t), Y) + \epsilon)d\mu(t) < 1\}$$ 

$$< \epsilon + \inf \{c > 0 : \int_0^1 \phi(c^{-1}\text{dist}(f(t), Y) + \|f(t) - f'(t)\| + \epsilon)d\mu(t) < 1\}.$$ 

$$< \epsilon + \inf \{c > 0 : \int_0^1 \phi(c^{-1}\text{dist}(f(t), Y) + 2\epsilon)d\mu(t) < 1\}.$$ 

Since $\epsilon$ is arbitrary, we have

$$\text{dist}(f, L_\Phi(u, Y)) < \inf \{c > 0 : \int_0^1 \phi(c^{-1}\text{dist}(f(t), Y))d\mu(t) < 1\}.$$ 

REMARK 2.1. For $f \in L_\Phi(u, X)$,

$$\|f\|_\Phi = \inf \{c > 0 : \int_0^1 \phi(\|f(t)\|/c)d\mu(t) < 1\} = c_0$$

such that $\int_0^1 \phi(\|f(t)\|/c_0)d\mu(t) = 1.$
COROLLARY 2.1. Let $Y$ be a closed subspace of $X$. To an element $f$ of $L_\phi(\mu, X)$, $g$ of $L_\phi(\mu, Y)$ is a best approximant of $f$ in $L_\phi(\mu, Y)$ if and only if $g(t)$ is a best approximant of $f(t)$ in $Y$.

PROOF. Let $g(t)$ be a best approximant of $f(t)$ in $Y$. This means that

$$\|f(t) - g(t)\| < \|f(t) - y\|$$

for all $t$ and for all $y \in Y$.

It follows that for any $h \in L_\phi(\mu, Y)$

$$\|f(t) - g(t)\| < \|f(t) - h(t)\|$$

for all $t$.

Since $\phi$ is increasing, we have

$$\phi^{-1}(\|f(t) - g(t)\|) < \phi^{-1}(\|f(t) - h(t)\|)$$

for any $c > 0$.

Then

$$\int_0 1 \phi^{-1}(\|f(t) - g(t)\|) \, d\mu(t) < \int_0 1 \phi^{-1}(\|f(t) - h(t)\|) \, d\mu(t).$$

Therefore

$$\inf \{ c > 0 : \int_0 1 \phi^{-1}(\|f(t) - g(t)\|) \, d\mu(t) < 1 \} < \inf \{ c > 0 : \int_0 1 \phi^{-1}(\|f(t) - h(t)\|) \, d\mu(t) \leq 1 \}$$

or

$$\|f - g\|_\phi < \|f - h\|_\phi$$

for all $h \in L_\phi(\mu, Y)$.

Conversely, let $g$ be a best approximant of $f$ in $L_\phi(\mu, Y)$, then

$$\text{dist}(f, L_\phi(\mu, Y)) = \|f - g\|_\phi.$$  By Lemma 2.2 and the previous remark, we have

$$\|f - g\|_\phi = \inf \{ c > 0 : \int_0 1 \phi^{-1}(\text{dist}(f(t), Y)) \, d\mu(t) < 1 \} = c_0$$

such that

$$\int_0 1 \phi^{-1}(\text{dist}(f(t), Y)) \, d\mu(t) = 1.$$  Hence

$$\int_0 1 \phi^{-1}(\text{dist}(f(t), Y)) \, d\mu(t) = 1.$$  

Since $\phi$ is strictly increasing and $\phi(c_0^{-1}(\|f(t) - g(t)\|) > \phi(c_0^{-1}(\text{dist}(f(t), Y)))$

then

$$\|f(t) - g(t)\| = \text{dist}(f(t), Y).$$

Now we prove the main theorem of this paper.

THEOREM 2.1. Let $Y$ be a closed subspace of $X$, then the following are equivalent:
(1) \( L_\phi(\mu, Y) \) is proximinal in \( L_\phi(\mu, X) \)

(ii) \( L^1(\mu, Y) \) is proximinal in \( L^1(\mu, X) \).

**Proof.** (ii) \( \Rightarrow \) (i). Let \( f \in L_\phi(\mu, X) \), then by Lemma 2.1 \( f \in L^1(\mu, X) \). By the assumption, there exists \( g \in L^1(\mu, Y) \) such that

\[
\| f - g \|_1 < \| f - h \|_1 \quad \text{for every } h \in L^1(\mu, Y).
\]

By Lemma 2.10 [3], we have

\[
\| f(t) - g(t) \| < \| f(t) - y \| \quad \text{for all } t \text{ and for all } y \in Y.
\]

Hence by Corollary 2.1 it follows that \( g \) is a best approximant of \( f \) in \( L_\phi(\mu, Y) \).

Conversely: (i) \( \Rightarrow \) (ii). Define a map

\[
J: L^1(\mu, X) \rightarrow L_\phi(\mu, X) \quad \text{by} \quad J(f) = \hat{f} \text{ where } \hat{f}(t) = \phi^{-1}(\| f(t) \|) f(t)
\]

if \( f(t) \neq 0 \), and zero otherwise. Then for \( c = 1 \)

\[
\int_0^1 \phi(\left| \phi^{-1} f(t) \right|) \mu(t) = \int_0^1 \phi(\left| \phi^{-1}(\| f(t) \|) f(t) \right|) \mu(t) = \int_0^1 \phi(\| f(t) \|) \mu(t) < \infty
\]

for all \( f \in L^1(\mu, X) \). Hence \( J(f) \in L_\phi(\mu, X) \). Since \( \phi \) is strictly increasing, it follows that \( J \) is (1-1). To show that \( J \) is onto, let \( g \in L_\phi(\mu, X) \), then take

\[
f(t) = \phi(\left| \frac{g(t)}{|g(t)|} \right|) g(t)
\]

if \( g(t) \neq 0 \) and zero otherwise. Clearly \( f \in L^1(\mu, X) \) and

\[
J(f) = \phi^{-1}(\left| f(t) \right|) f(t) = \phi^{-1}(\phi(\left| g(t) \right|) \phi(\left| g(t) \right|)) \phi(\left| g(t) \right|) = g(t).
\]

Thus \( J \) is onto. Now let \( f \in L^1(\mu, X) \), then \( \hat{f} \in L_\phi(\mu, X) \). By assumption there exists \( g \in L_\phi(\mu, Y) \) such that
||\hat{f} - \hat{g}||_\phi < ||\hat{f} - \hat{h}||_\phi \text{ for all } \hat{h} \in L_\phi(u, Y),

then by Corollary 2.1 we have


\begin{align*}
||\hat{f}(t) - \hat{g}(t)||_\phi < ||\hat{f}(t) - \gamma||_\phi \text{ for all } \gamma \in Y \text{ or } \gamma \in \phi^{-1}(x) \\
||f(t) - g(t)||_\phi < ||f(t) - \phi^{-1}(g(t)||_\phi < ||f(t) - \phi^{-1}(g(t)||_\phi < ||f(t) - \phi^{-1}(g(t)||_\phi \text{ for all } \gamma, t \in Y.
\end{align*}

Using the facts that \(||g(t)||_\phi < 2||f(t)||_\phi\) since \(0 \in Y\) and \(\phi^{-1}(2||f(t)||_\phi) < \phi^{-1}(||f(t)||_\phi)\) we can show that \(w \in L^1(u, Y)\) as follows

\begin{align*}
||w(t)||_\phi &= \phi^{-1}(||f(t)||_\phi) \\
&< \phi^{-1}(2||f(t)||_\phi) \\
&< 2||f(t)||_\phi \\
&< ||f(t)||_\phi.
\end{align*}

Now take any \(h \in L^1(u, Y)\) then

\begin{align*}
\phi^{-1}(||f(t)||_\phi) \text{ for all } t
\end{align*}

Hence

\begin{align*}
||f(t) - w(t)||_\phi < ||f(t) - \frac{||f(t)||_\phi}{\phi^{-1}(||f(t)||_\phi)} \phi^{-1}(||f(t)||_\phi) h(t)||_\phi
\end{align*}

\begin{align*}
= ||f(t) - h(t)||_\phi \text{ for all } t \text{ and for all } h \in L^1(u, Y), \text{ so } L^1(u, Y) \text{ is proximinal in } L^1(u, X).
\end{align*}

As a corollary,

COROLLARY 2.2. If \(Y\) is a reflexive subspace of \(X\), then \(L_\phi(u, Y)\) is proximinal in \(L_\phi(u, X)\).

PROOF. It follows from the main theorem and Theorem 1.2 in Kahall [5].

THEOREM 2.2. Let \(Y\) be a proximinal subspace of \(X\). Then for every simple function \(f \in L_\phi(u, X), \ P(f, L_\phi(u, Y))\) is not empty.

PROOF. Let \(f = \sum_{i=1}^{n} x_i y_i\) be a simple function in \(L_\phi(u, X)\), where \(A_i\) are disjoint measurable sets in \([0,1]\). Set \(g = \sum_{i=1}^{n} \chi_i y_i\), where \(y_i \in P(x_i, Y)\). Let \(h\) be any
element in $L_\phi(\mu,Y)$, then
\[
\|f - h\|_\phi = \inf\{c > 0 : \int_0^1 \phi\left(\left\|c^{-1}(f(t) - h(t))\right\|\right) \, d\mu(t) < 1\}
\]
\[
= \inf\{c > 0 : \sum_{i=1}^n \phi\left(\left\|c^{-1}(f(t) - h(t))\right\|\right) \, d\mu(t) < 1\}
\]
\[
= \inf\{c > 0 : \int_0^1 \phi\left(\left\|c^{-1}(f(t) - h(t))\right\|\right) \, d\mu(t) < 1\}
\]
\[
= \inf\{c > 0 : \int_0^1 \phi\left(\left\|c^{-1}(f(t) - g(t))\right\|\right) \, d\mu(t) < 1\}
\]
\[
= \inf\{c > 0 : \sum_{i=1}^n \phi\left(\left\|c^{-1}(f(t) - g(t))\right\|\right) \, d\mu(t) < 1\}
\]
\[
= \inf\{c > 0 : \phi\left(\left\|c^{-1}(f(t) - g(t))\right\|\right) \, d\mu(t) < 1\}
\]
\[
= \inf\{c > 0 : \phi\left(\left\|c^{-1}(f(t) - g(t))\right\|\right) \, d\mu(t) < 1\}
\]

Hence $g \in P(f, L_\phi(\mu,Y))$.

**Theorem 2.3.** Let $Y$ be a closed subspace of $X$. If $L_\phi(\mu,Y)$ is proximinal in $L_\phi(\mu,X)$, then $Y$ is proximinal in $X$.

**Proof.** From Theorem 2.1, $L_\phi(\mu,Y)$ proximinal in $L_\phi(\mu,X)$ implies that $L_1(\mu,Y)$ is proximinal in $L_1(\mu,X)$. By Theorem 1.1 [2] this also implies that $L_\omega(\mu,Y)$ is proximinal in $L_\omega(\mu,X)$. For $x \in X$, define $f_x : \Omega \to X$ by $f_x(t) = x$ for all $t \in \Omega$. It is clear that $f_x \in L_\omega(\mu,X)$ for every $x \in X$, so there exists $h \in L_\omega(\mu,Y)$ such that
\[
\|f_x - h\|_\omega \leq \|f_x - w\| \quad \text{for every } w \in L_\omega(\mu,Y).
\]
In particular, take $w = f_y$, so
\[
\|f_x - h\|_\omega \leq \|f_x - f_y\|_\omega \quad \text{for every } y \in Y
\]
\[
= \|x - y\| \quad \text{for every } y \in Y.
\]
But 
\[
\|x - h(t)\| = \|f_x(t) - h(t)\|
\]
\[
\leq \|f_x - h\|
\]
\[
= \|f_x - f_y\|
\]
\[
= \|x - y\| \quad \text{for every } y \in Y.
\]
Hence every $t \in [0,1]$ gives a best approximant of $x$ in $Y$. Therefore $Y$ is proximinal in $X$.

The next theorem needs the following definitions:
DEFINITION 2.1. The subspace $Y$ is called $\phi$-summand of $X$ if there is a bounded projection $Q: X \to Y$ such that

$$\phi(\|x\|) = \phi(\|Q(x)\|) + \phi(\|(I-Q)(x)\|)$$

for all $x \in X$. Where $I$ is the identity map on $X$.

DEFINITION 2.2. The subspace $Y$ is called $\phi$-complemented in $X$ if there is a closed subspace $Z$ in $X$ such that $X = Y + Z$ and the projection $P: X \to Z$ is a contractive projection.

THEOREM 2.4. If $Y$ is $\phi$-complemented in $X$, the $L_\Phi (\mu, Y)$ is proximinal in $L_\Phi (\mu, X)$.

PROOF. Let $X = Y \oplus Z$, $P: X \to Z$ be a contractive projection from $X$ onto $Z$. Hence $x = (L-P)x + P(x)$, $\|P(x)\| < \|x\|$. For $f \in L_\Phi (\mu, X)$, set $f_1 = (I-P)f$, $f_2 = \phi$. Let $P: L_\Phi (\mu, X) \to L_\Phi (\mu, Z)$ and

$$p(f) = pof = f_2$$

for all $f \in L_\Phi (\mu, X)$. Then $L_\Phi (\mu, Z)$ is a contractive projection onto $L_\Phi (\mu, Z)$ and $L_\Phi (\mu, Y) = L_\Phi (\mu, \mu) \oplus L_\Phi (\mu, Z)$. Hence $L_\Phi (\mu, Y)$ is $\phi$-complemented in $L_\Phi (\mu, Y)$. By Lemma 1.4 [2] $L_\Phi (\mu, Y)$ is proximinal in $L_\Phi (\mu, Y)$.

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