ON POINT - DISSIPATIVE SYSTEMS OF DIFFERENTIAL EQUATIONS WITH QUADRATIC NONLINEARITY

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(Received April 25, 1990)

ABSTRACT. The system \( x' = Ax + f(x) \) of nonlinear vector differential equations, where the nonlinear term \( f(x) \) is quadratic with orthogonality property \( x^T f(x) = 0 \) for all \( x \), is point-dissipative if \( u^T A u < 0 \) for all nontrivial zeros \( u \) of \( f(x) \).

KEY WORDS AND PHRASES. Point-dissipative, quadratic nonlinearity, symmetric matrices, commutative but generally non-associative algebra.

AMS 1980 Mathematics Subject Classification Number 34.

I. INTRODUCTION.

We are concerned with a class of nonlinear vector equations of the form
\[
x' = Ax + f(x)
\]  
(1.1)
where the nonlinear term \( f(x) \) is quadratic of the form
\[
f(x) = \begin{bmatrix}
x^T C_1 x \\
\vdots \\
x^T C_n x
\end{bmatrix}
\]
The \( n \times n \) matrices \( \{C_i\} \) are symmetric with the orthogonality property
\[
x^T f(x) = 0
\]  
(1.2)
for all \( x \).

We are interested in investigating the conditions on the \( n \times n \) matrix \( A \) and \( f(x) \) so that the system is point-dissipative, i.e., there is a bounded region which every trajectory of the system eventually enters and remains within.
II. DEFINITIONS.

For each vector $\alpha^T = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, we define the matrix $C(\alpha)$ as follows:

$$C(\alpha) = \sum_{i=1}^{n} \alpha_i C_i - \frac{A + A^T}{2}$$  \hspace{1cm} (2.1)

The mapping $xQy: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, where

$$xQy = \begin{pmatrix} x^TC_1y \\ \vdots \\ x^TC_ny \end{pmatrix}$$  \hspace{1cm} (2.2)

can be regarded as a commutative multiplication in $\mathbb{R}^n$. Note that

$$f(x) = xQx$$

and the quadratic formula

$$f(c_1x) = c_1 xQc_1x = c_1^2 xQx = c_1^2 f(x)$$

is true for all vectors $u_1, u_2, u_3$ and all scalars $c_1, c_2, c_3$.

In addition to the standard vector addition and scalar multiplication in $\mathbb{R}^n$, this multiplication $xQy$ gives the vector space $\mathbb{R}^n$ an additional structure of a commutative but generally non-associative algebra $B$. The algebra $B$ is determined uniquely by the symmetric $n \times n$ matrices $\{C_i\}$. This algebra has been studied by many, especially by Markus [1], Gerber [2], and Frayman [3].

Some algebraic properties of this algebra $B$ will be used to investigate the conditions for point-dissipativeness of the system (1.1). We are specially interested in the concepts of nilpotent and idempotent elements of the algebra $B$. A nilpotent element $v \neq 0$ satisfies $f(v) = vQv = 0$, while an idempotent element $v \neq 0$ satisfies $f(v) = vQv = v$. It has been proved [3] that in any such algebra $B$ (with or without the orthogonality property $x^T(xQx) = 0$ for all $x$) generated by any given $n$ symmetric matrices $\{C_i\}$, there exists at least one of these elements.

In our case, because of the orthogonality property (1.2), there cannot exist an idempotent element in the algebra $B$. For, if $u \neq 0$ is an idempotent, then

$$0 = u^Tf(u) = u^T(uQu) = u^Tu = \|u\|^2 \neq 0$$

gives us a contradiction. Hence, there must exist at least one nilpotent element in the algebra $B$. Again by (2.3), a scalar multiple of a nilpotent is also a nilpotent. Hence, the nonlinear quadratic term $f(x)$ in (1.1) has at least one 1-dimensional subspace of zeros.

As an example of system (1.1) with orthogonality property (1.2), we cite the Lorenz system:

$$x' = Ax + f(x)$$  \hspace{1cm} (2.4)

where
\[
A = \begin{pmatrix}
-a & a & 0 \\
r & -1 & 0 \\
0 & 0 & -b \\
\end{pmatrix}, \quad a > 0, \ r > 0, \ b > 0
\]

\[
f(x) = \begin{pmatrix}
0 \\
-xz \\
xy
\end{pmatrix}
\]

III. LEMMA 1. If there exists an \( \alpha \) so that \( C(\alpha) \) is positive definite, then the system

\[
x' = Ax + f(x)
\]

is point-dissipative.

The condition on \( A \) and \( f(x) \) which guarantees the existence of such an \( \alpha \) is the topic of our main theorem.

PROOF OF LEMMA 1. Suppose that there exists a vector \( \alpha \) such that the matrix \( C(\alpha) \) is positive definite. To show that the system (1.1) is point-dissipative, we need to exhibit a bounded region \( G \) so that the (positive) trajectory of each solution of (1.1) eventually enters and remains in \( G \). We construct a Lyapunov function of the form

\[
V(x) = \frac{1}{2} (x - \alpha)^T (x - \alpha)
\]

for which

\[
\dot{V}(x) = \alpha^T Ax - x^T C(\alpha)x
\]

Since the quadratic term \( x^T C(\alpha)x \) dominates the linear term \(-\alpha^T Ax\), the set

\[
S = \{ x \mid \dot{V}(x) \geq 0 \}
\]

is bounded. Hence we can choose \( r_0 > 0 \), sufficiently large, so that the level set (sphere) \( V(x) = r_0 \) contains in its interior the bounded set \( S \). We choose the interior of the sphere \( V(x) = r_0 \) to be our bounded region \( G \). Let \( P_0 \) be a point outside of \( G \) and \( \Phi(t, P_0) \) be the solution of (1.1) with \( \Phi(0, P_0) = P_0 \). Let \( V(x) = r_1 \) be the level set of \( V(x) \) passing through \( P_0 \). Clearly \( r_1 > r_0 \). Let \( H \) be the annular closed region formed by the two concentric spheres \( V(x) = r_1 \) and \( V(x) = r_0 \). Since the bounded set \( S \) lies inside the sphere \( V(x) = r_0 \), \( \dot{V}(x) < 0 \) on \( H \). Therefore, \( V(\Phi(t, P_0)) \) is a decreasing function of \( t \) on \( H \). Hence, the trajectory of \( \Phi(t, P_0) \) must enter the sphere \( V(x) = r_1 \) and cannot go outside of the sphere \( V(x) = r_1 \) at any time \( t > 0 \). Suppose that the trajectory of \( \Phi(t, P_0) \) cannot enter the region \( G \). Then it must remain in \( H \) for all time \( t \geq 0 \). It must have a limit point \( P \) in \( H \). By using standard proof we can show that \( \dot{V}(P) = 0 \) which gives us a contradiction as \( \dot{V}(x) < 0 \) on \( H \). Hence, the trajectory of \( \Phi(t, P_0) \) must eventually enter the bounded region \( G \) and cannot go out of \( G \) by the decreasing property of \( V(\Phi(t, P_0)) \) and therefore must remain in \( G \).

IV. THEOREM. For \( n = 2, 3 \), the system \( x' = Ax + f(x) \) is point-dissipative if and only if \( u^T Au < 0 \) for all nontrivial zeros \( u \) of \( f(x) \).

For \( n = 2 \), the theorem has already been proved by Bose and Reneke [1]. Hence we will give the proof for \( n = 3 \). In order to prove the theorem, all we need to show is that the condition \( u^T Au < 0 \) for all nontrivial zeros of \( f(x) \) implies that there exists a vector \( \alpha \) such that the matrix \( C(\alpha) \) is positive definite. Hence, by Lemma 1, the theorem will be proved.
We also need the following definitions and lemmas:

**DEFINITION 1.** Let \( Z \) be the set of all zeros of \( f(x) \). That is \( Z \) contains the zero vector and all the nilpotents of the algebra \( B \).

**DEFINITION 2.** \( S(u, v) \) is the 2-dimensional subspace of \( \mathbb{R}^3 \) generated by two linearly independent vectors \( u \) and \( v \).

**DEFINITION 3.** \( S(u) \) is the 1-dimensional subspace of \( \mathbb{R}^3 \) generated by a nontrivial vector \( u \).

**LEMMA 2.** If \( u \) is a zero of \( f(x) \), then \( u^T q x \) is orthogonal to \( u \) for all \( x \).

**LEMMA 3.** If \( u, v \) are two linearly independent zeros of \( f(x) \), then \( S(u, v) \subseteq Z \) if and only if \( u^T q v = 0 \).

**PROOF OF LEMMA 2.** Suppose that \( u \) be a zero of \( f(x) \). Then by using the quadratic formula (2.3) and the orthogonality relations \( (u + x)^T f(u + x) = 0 \), \( (u - x)^T f(u - x) = 0 \), we can show that \( u^T (u^T q x) = 0 \), for all \( x \).

**PROOF OF LEMMA 3.** Let \( u \) and \( v \) be two linearly independent zeros of \( f(x) \).

Suppose that \( u^T q v = 0 \). Then \( f(c_1 u + c_2 v) = c_1^2 u^T q u + 2 c_1 c_2 u^T q v + c_2^2 v^T q v = 0 \) implies that \( c_1 u + c_2 v \) is in \( Z \) for any two scalars \( c_1 \) and \( c_2 \). Hence, \( S(u, v) \subseteq Z \). Conversely, suppose that \( S(u, v) \subseteq Z \). Then \( u + v \) is in \( Z \) and

\[
0 = f(u + v) = u^T q u + 2 u^T q v + v^T q v = 2 u^T q v \implies u^T q v = 0.
\]

Let \( u_1, u_2, u_3 \) be a basis of \( \mathbb{R}^3 \), then for any vector \( x = d_1 u_1 + d_2 u_2 + d_3 u_3 \),

\[
x^T C(\alpha) x = \alpha^T f(x) - x^T A x = d^T \hat{C}(\alpha) d
\]

where \( d^T = (d_1, d_2, d_3) \) and the matrix \( \hat{C}(\alpha) = ((c_{ij})) \) with

\[
c_{ij} = \alpha^T (u_i^T q u_j) - u_i^T A u_j, \quad i, j = 1, 2, 3,
\]

\[
c_{ij} = c_{ji}
\]

Hence, in order to show that the matrix \( C(\alpha) \) is positive definite for some \( \alpha \), all we need to show is that the matrix \( \hat{C}(\alpha) \) is positive definite for some \( \alpha \).

**PROOF OF THE THEOREM.** That the condition "\( u^T A u < 0 \) for all nontrivial \( u \) in \( Z \)" is necessary follows from (4.1). Hence we need only to show that it is also sufficient.

The proof of the theorem depends on the nature of the set \( Z \) of all zeros of \( f(x) \).

We need to consider the following cases:

Case 1. (a) \( Z \) contains 3 linearly independent vectors with **three** 2-dimensional subspace of zeros.

(b) \( Z \) contains 3 linearly independent vectors with **two** 2-dimensional subspace of zeros.

(c) \( Z \) contains 3 linearly independent vectors with **one** 2-dimensional subspace of zeros.

(d) \( Z \) contains 3 linearly independent vectors with **no** 2-dimensional subspace of zeros.
Case 2. (a) $Z$ contains 2 linearly independent vectors with one 2-dimensional subspace of zeros.
(b) $Z$ contains 2 linearly independent vectors with no 2-dimensional subspace of zeros.

Case 3. $Z$ contains only one linearly independent vector.

Case 1(a) cannot happen. For suppose that $u_1, u_2, u_3$ be three linearly independent vector in $Z$ so that $Z = S(u_1, u_2) \cup S(u_1, u_3) \cup S(u_2, u_3)$. Then by lemma 3

$$u_i Qu_j = 0, \text{ for all } i, j = 1, 2, 3.$$ Hence, for any vector $x = c_1 u_1 + c_2 u_2 + c_3 u_3$, $f(x) = \sum_{i,j=1}^{3} c_i c_j u_i Qu_j = 0$, implies that $f(x) = 0$, for all $x$.

Case 1(b) also cannot happen. For suppose that $u_1, u_2, u_3$ be three linearly independent vectors in $Z$ so that $Z = S(u_1, u_2) \cup S(u_1, u_3) \cup S(u_3)$. Then by lemma 3, $u_i Qu_i = 0$, for $i = 1, 2, 3$, $u_1 Qu_2 = 0$, $u_1 Qu_3 = 0$ but $u_2 Qu_3 \neq 0$. Now $f(u_1 + u_2 + u_3) = 2u_2 Qu_3$ and $(u_1 + u_2 + u_3)^T f(u_1 + u_2 + u_3) = 0$ implies that $u_1^T (u_2 Qu_3) = 0$. This implies by lemma 2 that $u_2 Qu_3$ is orthogonal to each of the basis vector $u_1, u_2, u_3$ and hence $u_2 Qu_3 = 0$, contradicting our hypothesis.

Case 1(c). Let $u_1, u_2, u_3$ be three linearly independent vectors in $Z$ so that $Z = S(u_1, u_2) \cup S(u_3)$. Here $u_i Qu_i = 0$, $i = 1, 2, 3$, $u_1 Qu_2 = 0$ but $u_1 Qu_3 \neq 0$, $u_2 Qu_3 \neq 0$.

By hypothesis of the theorem

$$(c_1 u_1 + c_2 u_2)^T A(c_1 u_1 + c_2 u_2) = \sum_{i,j=1}^{2} c_i c_j u_i^T A u_j$$

$$=(c_1, c_2) \begin{pmatrix} u_1^T A u_1 & u_1^T A u_2 \\ u_2^T A u_1 & u_2^T A u_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} < 0$$

for all $(c_1, c_2) \neq (0, 0)$. That is

$$\begin{pmatrix} -u_1^T A u_1 & -u_1^T A u_2 \\ -u_2^T A u_1 & -u_2^T A u_2 \end{pmatrix}$$

is positive definite.

Again $u_1 Qu_3$ and $u_2 Qu_3$ must be linearly independent. For suppose that $c_1 (u_1 Qu_3) + c_2 (u_2 Qu_3) = 0$, for some scalars $c_1$ and $c_2$. Taking inner product respectively with $u_1$ and $u_2$ and using lemma 2, we get

$$c_2 u_1^T (u_2 Qu_3) = 0$$
$$c_1 u_2^T (u_1 Qu_3) = 0.$$ Now $u_1^T (u_2 Qu_3) = 0$ implies by lemma 2 that $u_2 Qu_3$ is orthogonal to each of the basis vector $u_1, u_2, u_3$ and hence $u_2 Qu_3 = 0$ contradicting our hypothesis that $u_2 Qu_3 \neq 0$. 
Therefore $u_1^T(u_2Qu_3) \neq 0$, implying that $c_2 = 0$. Similarly $c_1 = 0$. Hence $u_1Qu_3$ and $u_2Qu_3$ are linearly independent. We can choose a vector $\alpha$ such that

$$\alpha^T(u_1Qu_3) - u_1^T Au_3 = 0$$

$$\alpha^T(u_2Qu_3) - u_2^T Au_3 = 0.$$  

For such a choice of $\alpha$, the matrix $\hat{C}(\alpha)$ becomes

$$\hat{C}(\alpha) = \begin{pmatrix} -u_1^T Au_1 & -u_1^T Au_2 & 0 \\ -u_2^T Au_1 & -u_2^T Au_2 & 0 \\ 0 & 0 & -u_3^T Au_3 \end{pmatrix}$$

which is positive definite.

Case 1(d). Let $u_1, u_2, u_3$ be three linearly independent vectors in $Z$ so that $Z = S(u_1) \cup S(u_2) \cup S(u_3)$. Here $u_iQu_j = 0$, if $i = j$ and $u_iQu_j \neq 0$, if $i \neq j$. As in case 1(c), we can show that $u_1Qu_2$, $u_1Qu_3$, $u_2Qu_3$ are linearly independent. Hence we can choose a vector $\alpha$ such that

$$c_{12} = \alpha^T(u_1Qu_2) - u_1^T Au_2 = 0$$

$$c_{13} = \alpha^T(u_1Qu_3) - u_1^T Au_3 = 0$$

$$c_{23} = \alpha^T(u_2Qu_3) - u_2^T Au_3 = 0.$$  

For such a choice of $\alpha$, the matrix $\hat{C}(\alpha)$ becomes

$$\hat{C}(\alpha) = \begin{pmatrix} -u_1^T Au_1 & 0 & 0 \\ 0 & -u_2^T Au_2 & 0 \\ 0 & 0 & -u_3^T Au_3 \end{pmatrix}$$

which is positive definite.

Case 2(a). Let $u_1, u_2$ be two linearly independent vectors in $Z$ such that $Z = S(u_1, u_2)$. We can assume that $u_1$ and $u_2$ are two unit vectors orthogonal to each other. Let $u_3$ be a unit vector such that $u_1, u_2, u_3$ form an orthonormal basis of $R^3$. Here,

$$u_1Qu_1 = u_1Qu_2 = u_2Qu_2 = 0, \quad u_3Qu_3 \neq 0$$

Since $u_1Qu_3$ is orthogonal to $u_1$ and $u_2Qu_3$ is orthogonal to $u_3$, we can write

$$u_1Qu_3 = t_1u_2 + t_2u_3, \quad u_3Qu_3 = p_1u_1 + p_2u_2.$$  

Using the orthogonality property $(u_1 + u_3)^Tf(u_1 + u_3) = 0$, we can show that $p_1 = -2t_2$. Hence, $u_3Qu_3 = -2t_2u_1 + p_2u_2$. $(t_2, p_2) \neq (0, 0)$. Similarly we can show that

$$u_2Qu_3 = -t_1u_1 - \frac{1}{2}p_2u_3.$$  

Now, in this case $t_1 = 0$. For $t_1 \neq 0$ implies that

$$f \left( -\frac{p_2}{2t_1}u_1 - \frac{t_2}{t_1}u_2 + u_3 \right) = 0.$$  

Since $-\frac{p_2}{2t_1}u_1 - \frac{t_2}{t_1}u_2 + u_3$ is not in $Z$, we get a
contradiction. Hence, $u_1 Qu_3 = t_2 u_3$, $u_2 Qu_3 = -\frac{1}{2} p_2 u_3$, $u_3 Qu_3 = -2t_2 u_1 + p_2 u_2$. As in case 1(c),

\[
\begin{pmatrix}
-u_1^T A u_1 & -u_1^T A u_2 \\
-u_2^T A u_1 & -u_2^T A u_2 \\
-u_3^T A u_1 & -u_3^T A u_2 \\
\end{pmatrix}
\]

is positive definite. Taking $\alpha = -\frac{1}{2} r t_2 u_1 + r p_2 u_2$, where $r > 0$, to be chosen suitably, the matrix $\hat{C}(\alpha)$ becomes

\[
\hat{C}(\alpha) = \begin{pmatrix}
-u_1^T A u_1 & -u_1^T A u_2 & -u_1^T A u_3 \\
-u_2^T A u_1 & -u_2^T A u_2 & -u_2^T A u_3 \\
-u_3^T A u_1 & -u_3^T A u_2 & r(t_2^2 + p_2^2) - u_3^T A u_3 \\
\end{pmatrix}
\]

Here, $\det \hat{C}(\alpha) = r(t_2^2 + p_2^2) + \delta$ where $\delta$ is a constant (independent of $r$). Clearly we can choose $r > 0$, sufficiently large, to make $r(t_2^2 + p_2^2) - u_3^T A u_3 > 0$ and $\det \hat{C}(\alpha) > 0$. In other words we can choose a vector $\alpha$ such that $\hat{C}(\alpha)$ is positive definite.

Case 2(b). Let $u_1, u_2$ be two linearly independent unit vectors in $Z$ so that $Z = S(u_1) \cup S(u_2)$. Let $u_3$ be a unit vector orthogonal to $S(u_1, u_2)$. Then $u_1, u_2, u_3$ form a basis of $R^3$ with $u_3^T u_1 = 0, u_3^T u_2 = 0$. Here, $u_1 Qu_2 \neq 0$ and $u_3 Qu_3 \neq 0$. As in previous cases we can show using lemma 2 and the orthogonality property of $f(x)$ that

\[
u_1 Qu_2 = s_2 u_3, s_2 \neq 0, u_1 Qu_3 = -(t_1 u_1^T u_2) u_1 + t_1 u_2 + t_2 u_3,
u_3 Qu_3 = -(2t_2 + q_2 u_1^T u_2) u_1 + q_2 u_2, \text{ and}
\]

\[
u_2 Qu_3 = -(t_1 + \frac{s_2}{1 - (u_1^T u_2)^2}) u_1 + (t_1 + \frac{s_2}{1 - (u_1^T u_2)^2}) (u_1^T u_2) u_2 + \frac{1}{2} \left(2t_2 u_1^T u_2 \left(1 - (u_1^T u_2)^2\right)\right) u_2
\]

Now in this case $t_2 = 0$ implies $t_1 = 0$. For, if $t_2 = 0$, then $f \left(\frac{q_2}{2} u_1 \cdot t_1 u_3\right) = 0$ implies that $t_1 = 0$. Hence $t_1 \neq 0$ implies that $t_2 \neq 0$.

In order to prove case 2(b), we also need the following two results:

(i) If $t_1 \neq 0$, then $t_1 + \frac{s_2}{1 - (u_1^T u_2)^2} = 0$

(ii) If $t_1 = 0$, then $2t_2 u_1^T u_2 - q_2 \left(1 - (u_1^T u_2)^2\right) \neq 0$

To prove result (i), suppose that $t_1 \neq 0$. We need to show that the vectors $u_1 Qu_2, u_1 Qu_3,$
$u_2Qu_3$ are linearly dependent. Suppose that they are linearly independent. Then 

$$u_3Qu_3 = c_1(u_1Qu_2) + c_2(u_1Qu_3) + c_3(u_2Qu_3)$$

for some $(c_1, c_2, c_3) \neq (0, 0, 0)$. Now

$$f(\frac{1}{2}c_2u_1 + \frac{1}{2}c_3u_2 - u_3) = \frac{1}{2}c_2c_3 + c_1$$

$$(u_1Qu_2) = \frac{1}{2}c_2c_3 + c_1$$

$$(u_1Qu_2) = \frac{1}{2}c_2c_3 + c_1$$

Since $(\frac{1}{2}c_2u_1 + \frac{1}{2}c_3u_2 - u_3)^T f(\frac{1}{2}c_2u_1 + \frac{1}{2}c_3u_2 - u_3) = 0$, we have $\frac{1}{2}c_2c_3 + c_1 = 0$. This in turn implies that $f(\frac{1}{2}c_2u_1 + \frac{1}{2}c_3u_2 - u_3) = 0$ giving us a contradiction. Hence

$$c_1(u_1Qu_2) + c_2(u_1Qu_3) + c_3(u_2Qu_3) = 0,$$

for some $(c_1, c_2, c_3) \neq (0, 0, 0)$. That is

$$\left\{ \begin{array}{l}
    c_2 t_1 u_1^T u_2 + \left( t_1 + \frac{s_2}{1 - (u_1^T u_2)^2} \right) c_3 \\
    c_2 t_1 + \left( t_1 + \frac{s_2}{1 - (u_1^T u_2)^2} \right) (u_1^T u_2) c_3
\end{array} \right\} u_2$$

$$+ \left\{ c_1 s_2 + c_2 t_2 + \frac{1}{2} c_3 \left( 2t_2 u_1^T u_2 - q_2 (1 - (u_1^T u_2)^2) \right) \right\} u_3 = 0$$

That is $(c_1, c_2, c_3) \neq (0, 0, 0)$ must be a solution of the linear system

$$c_2 t_1 \left( u_1^T u_2 \right) + \left( t_1 + \frac{s_2}{1 - (u_1^T u_2)^2} \right) c_3 = 0$$

$$c_2 t_1 + \left( t_1 + \frac{s_2}{1 - (u_1^T u_2)^2} \right) (u_1^T u_2) c_3 = 0$$

$$c_1 s_2 + c_2 t_2 + \frac{1}{2} \left( 2t_2 u_1^T u_2 - q_2 (1 - (u_1^T u_2)^2) \right) c_3 = 0$$

Now

$$\begin{vmatrix}
    t_1 u_1^T u_2 \\
    t_1 + \frac{s_2}{1 - (u_1^T u_2)^2} \\
    t_1 + \frac{s_2}{1 - (u_1^T u_2)^2}
\end{vmatrix}
= \begin{vmatrix}
    t_1 \left( t_1 + \frac{s_2}{1 - (u_1^T u_2)^2} \right) (u_1^T u_2)^2 - 1 \\
\end{vmatrix}$$

Since $u_1$ and $u_2$ are two linearly independent unit vectors, $|u_1^T u_2| < 1$ and therefore

$$(u_1^T u_2)^2 - 1 \neq 0.$$ 

If $t_1 + \frac{s_2}{1 - (u_1^T u_2)^2} \neq 0$, then $c_2 = c_3 = 0$. This in turn implies that $c_1 = 0$ contradicting our hypothesis that $(c_1, c_2, c_3) \neq (0, 0, 0)$. Hence $t_1 \neq 0$ implies that

$$t_1 + \frac{s_2}{1 - (u_1^T u_2)^2} = 0.$$

To prove result (ii), suppose that $t_1 = 0$. If $2t_2(u_1^T u_2) - q_2(1 - (u_1^T u_2)^2) = 0$, then

$$(2t_2 + q_2(u_1^T u_2))(u_1^T u_2) = q_2, u_1Qu_2 = s_2u_3, s_2 \neq 0, u_1Qu_3 = t_2u_3.$$
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\[ u_3 Q u_3 = -\frac{q_2}{u_1^T u_2} u_1 + q_2 u_2 \] (assuming \( u_1^T u_2 \neq 0 \)) and
\[ u_2 Q u_3 = \frac{s_2}{1 - (u_1^T u_2)^2} \left\{ -u_1 + (u_1^T u_2) u_2 \right\} \]

and \( f \left( -\frac{q_2}{u_1^T u_2} u_2 + \frac{2s_2}{1 - (u_1^T u_2)^2} u_3 \right) = 0 \). Since \( s_2 \neq 0 \), this implies a contradiction.

Therefore \( t_1 = 0 \) implies that \( 2t_2(u_1^T u_2) - q_2(1 - (u_1^T u_2)^2) \neq 0 \). In case \( u_1^T u_2 = 0 \), we can show that \( q_2 \neq 0 \).

To prove case 2(b), we will consider the following two subcases:

\((g)\) \( t_1 \neq 0 \), and
\((h)\) \( t_1 = 0 \).

Consider the subcase \((g)\) first. We have \( t_1 \neq 0 \), then by result (i) \( t_1 + \frac{s_2}{1 - (u_1^T u_2)^2} = 0 \).

For this subcase \( u_1 Q u_2 = s_2 u_3, s_2 \neq 0 \), \( u_1 Q u_3 = -(t_1 u_1^T u_2) u_1 + t_1 u_2 + t_2 u_3 \),
\( u_2 Q u_3 = (t_2(u_1^T u_2) - \frac{1}{2} q_2 (1 - (u_1^T u_2)^2)) u_3 \), \( u_3 Q u_3 = -(2t_2 + q_2 u_1^T u_2) u_1 + q_2 u_2 \).

Taking \( \alpha = k_1 u_1 + k_2 u_2 + k_3 u_3 \) the entries \( c_{ij} \) of the matrix \( \hat{C}(\alpha) \) becomes,
\[ c_{11} = -u_1^T A u_1, c_{22} = -u_2^T A u_2 \]
\[ c_{12} = \alpha^T u_1 Q u_2 - u_1^T A u_2 = s_2 k_3 - u_2^T A u_2 \]
\[ c_{13} = \alpha^T u_1 Q u_3 - u_1^T A u_3 = t_1 (1 - (u_1^T u_2)^2) k_2 + t_2 k_3 - u_2^T A u_3 \]
\[ c_{23} = \alpha^T u_2 Q u_3 - u_2^T A u_3 = (t_2 u_1^T u_2 - \frac{1}{2} (1 - (u_1^T u_2)^2)) k_3 - u_2^T A u_3 \]
\[ c_{33} = \alpha^T u_3 Q u_3 - u_3^T A u_3 = -2t_2 k_1 - (2t_2 u_1^T u_2 - q_2 (1 - (u_1^T u_2)^2)) k_2 - u_2^T A u_3 \]

We can choose \( k_3 \) so that \( c_{12} = \alpha^T u_1^T Q u_2 - u_1^T A u_2 = 0 \). For this \( k_3 \)
\[ c_{23} = \alpha^T u_2 Q u_3 - u_2^T A u_3 = \text{constant} = \delta \] (say). After choosing \( k_3 \), we can now choose \( k_2 \) so that \( c_{13} = \alpha^T u_1 Q u_3 - u_1^T A u_3 = 0 \). After choosing \( k_2 \) and \( k_3 \) in this way, we now choose
\[ k_1 = -\frac{1}{2} t_2 r, \text{ where } r > 0 \] to be chosen suitably. For such a choice of \( \alpha \), \( c_{33} = \alpha^T u_3 Q u_3 - u_3^T A u_3 = r + a \) where \( a \) is a constant independent of \( r \) and the matrix \( \hat{C}(\alpha) \) becomes
\[
\hat{C}(\alpha) = \begin{pmatrix}
-u_1^T A u_1 & 0 & 0 \\
0 & u_1^T A u_2 & \delta \\
0 & \delta & t_2^2 r + a 
\end{pmatrix}
\]

Clearly we can choose \( r > 0 \), sufficiently large to make \( \hat{C}(\alpha) \) positive definite. The subcase \((h)\) can be similarly disposed of, using the fact that \( t_1 = 0 \) implies
\[ 2t_2(u_1^T u_2) - q_2(1 - (u_1^T u_2)^2) \neq 0. \]
Case 3. Let \( u \) be a unit vector in \( Z \) so that \( Z \subseteq S(u) \). Let \( u, v, w \) be an orthonormal basis of \( \mathbb{R}^3 \). By our assumption \( vQv \neq 0 \) and \( wQw \neq 0 \). Using lemma 2 and the orthogonality property (1.2), we can write

\[
\begin{align*}
  uQv &= s_1 v + s_2 w \\
  uQw &= t_1 v + t_2 w \\
  vQv &= -2s_1 u + p w \\
  vQw &= -2t_2 u + q v \\
  wQw &= -(t_1 + s_2) u - \frac{1}{2} p v - \frac{1}{2} q w
\end{align*}
\]

We will solve this case by considering three subcases:

Subcase (a): \( D \) is of rank 2

Subcase (b): \( D \) is of rank 1

Subcase (c): \( D \) is of rank 0

We also need the following two results (i) and (ii):

(i) \( s_2 \neq 0 \) implies \( s_1 = 0 \)

(ii) \( s_1 \neq 0 \) implies \( s_2 = 0 \)

The result (i) can be proved as in case 2(b). For the result (ii), suppose that

\[ s_1 = 0 \text{ and } s_2 \neq 0. \]

Then \( f \left( \frac{1}{2} pu - s_2 v \right) = 0 \) implies a contradiction. Hence, \( s_1 = 0 \) implies \( s_2 = 0 \).

Now consider the subcase (a). The matrix \( D \) is non-singular. This implies by (i) and (ii) that \( s_1 t_2 \neq 0 \), otherwise we would get a row of zeros. We will like to show that the quadratic form \( x^T Dx \neq 0 \) for any \( x \neq 0 \). Suppose that there exists \( x^T = (x_1, x_2) \neq (0, 0) \) such that \( x^T Dx = 0 \). Since \( s_1 t_2 \neq 0 \), it follows that \( x_1 x_2 \neq 0 \).

Since \( D \) is non-singular, the transpose \( D^T = \begin{pmatrix} s_1 & t_1 \\ s_2 & t_2 \end{pmatrix} \) is also non-singular and

\[
D^T x = \begin{pmatrix} s_1 x_1 + t_1 x_2 \\ s_2 x_1 + t_2 x_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

Without loss of generality, suppose that \( s_2 x_1 + t_2 x_2 \neq 0 \). Then for any scalar

\[
c f(cu + x_1 v + x_2 w) = \left( 2c(s_2 x_1 + t_2 x_2) + x_1(x_1 p - x_2 q) \right) \frac{x_2}{x_1} v + w
\]

Since \( s_2 x_1 + t_2 x_2 \neq 0 \), we can choose the scalar \( c \), to make \( f(cu + x_1 v + x_2 w) = 0 \) contradicting the fact that \( x_1 x_2 \neq 0 \). Hence \( x^T Dx \neq 0 \) for any \( x \neq 0 \). Therefore by continuity,

\[
x^T Dx = x^T \begin{pmatrix} s_1 & t_1 + s_2 \\ t_1 + s_2 & 2 t_2 \end{pmatrix} x
\]

is either positive definite or negative definite. In either case

\[
s_1 t_2 - \frac{1}{4} (t_1 + s_2)^2 > 0 \quad (4.2)
\]

(4.2) also implies that \( s_1 \) and \( t_2 \) are of the same sign.

Taking \( \alpha = k_1 u + k_2 v + k_3 w \), the entries \( c_{ij} \) of the matrix \( \hat{C}(\alpha) \) becomes

\[
\begin{align*}
  c_{11} &= \alpha^T uQu - u^T Au = -u^T Au \\
  c_{12} &= \alpha^T uQv - u^T Av = s_1 k_2 + s_2 k_3 - u^T Av
\end{align*}
\]
\[ c_{13} = \alpha^T u Q_w - u^T A w = t_1 k_2 + t_2 k_3 - u^T A w \\
\]
\[ c_{22} = \alpha^T v Q_v - v^T A v = -2s_1 k_1 + p k_3 - v^T A v \\
\]
\[ c_{33} = \alpha^T w Q_w - w^T A w = -2t_2 k_1 + q k_2 - w^T A w \\
\]
\[ c_{23} = \alpha^T v Q_w - v^T A w = -(t_1 + s_2) k_1 - \frac{1}{2} p k_2 - \frac{1}{2} q k_2 - v^T A w \]

Since \( D \) is non-singular, we can choose \( k_2 \) and \( k_3 \) so that \( c_{12} = c_{13} = 0 \). Since \( s_1 \) and \( s_2 \) are of the same sign, we can choose \( k_1 \) with \( |k_1| \) sufficiently large to make \( c_{22} > 0, c_{33} > 0 \) and

\[
\begin{vmatrix}
  c_{22} & c_{23} \\
  c_{23} & c_{33}
\end{vmatrix} = \left\{ 4s_1 t_2 - (t_1 + s_2)^2 \right\} k_1^2 + k_1 d_1 + d_2 > 0
\]

where \( d_1 \) and \( d_2 \) are constants. Hence for such a choice of \( k_1, k_2, k_3 \) the matrix \( \hat{C}(\alpha) \) becomes

\[
\hat{C}(\alpha) = \begin{pmatrix}
  -u^T A u & 0 & 0 \\
  0 & c_{22} & c_{23} \\
  0 & c_{23} & c_{33}
\end{pmatrix}
\]

which is positive definite.

Now consider the subcase (b). Here rank \( D = 1 \). Without loss of generality we can assume that \( (t_1, t_2) \neq (0, 0) \). This implies that \( t_2 \neq 0 \), by property (1). Let \( (s_1, s_2) = k(t_1, t_2) \). This implies that \( k = t_1/t_2 \). For suppose that \( k \neq t_1/t_2 \). Then for any scalar \( c \),

\[ f(c u + t_2 v - t_1 w) = (2c(kt_2 - t_1) + t_2 p + t_1 q) (t_1 v + t_2 w) \]

Since \( kt_2 - t_1 \neq 0 \), we can choose the scalar \( c \) so that \( f(c u + t_2 v - t_1 w) = 0 \) implying that \( t_1 = t_2 = 0 \) contradicting our assumption. This also implies that \( t_2 p + t_1 q \neq 0 \). Hence

\[
D = \begin{pmatrix}
  t_1^2/t_2 & t_1 & t_2 \\
  t_1 & t_1 & t_2
\end{pmatrix}
\]

With this \( D \)

\[
u Q v = \frac{t_1^2}{t_2} v + t_1 w \\
u Q w = t_1 v + t_2 w
\]

\[
v Q v = -\frac{2t_1^2}{t_2} u + p w \\
w Q w = -2t_2 u + q v \\
v Q w = -2t_1 u - \frac{1}{2} p v - \frac{1}{2} q w
\]

Since \( v Q v \neq 0 \), we have \( (t_1, p) \neq (0, 0) \). Taking \( \alpha = \frac{1}{2} r_2 t_2 u + r_1 q v + r_1 p w \), where \( r_1 > 0, r_2 > 0 \) to be chosen suitably, the entries \( c_{ij} \) of the matrix \( \hat{C}(\alpha) \) becomes

\[
c_{11} = -u^T A u, \ c_{12} = \frac{r_1 t_1}{t_2} (t_1 q + t_2 p) - u^T A v, \ c_{13} = r_1(t_1 q + t_2 p) - u^T A w \\
c_{22} = r_2 t_1^2 + r_1 p^2 - v^T A v, \ c_{23} = t_1 t_2 r_2 - r_1 p q - v^T A w \\
c_{33} = r_2 t_2^2 + r_1 q^2 - w^T A w
\]
Now \( c_{11} = -u^T A u > 0 \),
\[
\begin{pmatrix}
  c_{11} & c_{12} \\
  c_{12} & c_{22}
\end{pmatrix} = r_2 (-u^T A u) t_1^2 + d_1 (r_1),
\]
where \( d_1 (r_1) \) is a quadratic in \( r_1 \) and \( \det \hat{C}(\alpha) = r_2 \left[ (-u^T A u) (t_1 q + t_2 p)^2 r_1 + d_2 \right] + d_3 (r_1) \), where \( d_2 \) is a constant and \( d_3 (r_1) \) is a cubic polynomial in \( r_1 \). Hence, if \( t_1 \neq 0 \), then we can choose \( r_1 > 0 \) large enough to make \(- (u^T A u) (t_1 q + t_2 p)^2 r_1 + d_2 > 0. \) After choosing such an \( r_1 > 0 \), we can choose \( r_2 > 0 \) sufficiently large to make \[
\begin{pmatrix}
  c_{11} & c_{12} \\
  c_{12} & c_{22}
\end{pmatrix} > 0 \text{ and } \det \hat{C}(\alpha) > 0.
\]
In other words we can choose \( \alpha \) so that \( \hat{C}(\alpha) \) is positive definite.

If \( t_1 = 0 \), then
\[
\begin{pmatrix}
  c_{11} & c_{12} \\
  c_{12} & c_{22}
\end{pmatrix} = (-u^T A u)p^2 r_1 + d_4,
\]
where \( d_4 \) is a constant and
\[
\det \hat{C}(\alpha) = r_2^2 \left[ (-u^T A u)p^2 r_1 + d_4 \right] + d_5 (r_1),
\]
where \( d_5 (r_1) \) is a quadratic in \( r_1 \). As before we can choose \( r_1 > 0 \) to make \((-u^T A u)p^2 r_1 + d_4 > 0\) and after choosing such an \( r_1 > 0 \), we can choose \( r_2 > 0 \) to make \( \det \hat{C}(\alpha) > 0 \). In other words we can choose an \( \alpha \) so that \( \hat{C}(\alpha) \) is positive definite.

Now consider the subcase (c). Here \( \text{rank } D = 0 \), which implies that \( s_1 = s_2 = t_1 = t_2 = 0 \).

Hence \( uQv = 0, uQw = 0, vQv = pw, p \neq 0, wQw = qv, q \neq 0, vQw = -\frac{1}{2} pv - \frac{1}{2} qw \) and \( f(qv + pw) = 0 \). Since \( pq \neq 0 \), this implies a contradiction. Hence, subcase (c) cannot happen.

This completes the proof.

For an example, the Lorenz system (2.4)
\[
x' = Ax + f(x),
\]
where
\[
A = \begin{pmatrix}
-a & a & 0 \\
r & -1 & 0 \\
0 & 0 & -b
\end{pmatrix}, \text{a > 0, r > 0, b > 0 and } f(x) = \begin{pmatrix}
-xz \\
xy
\end{pmatrix}
\]
is point dissipative. The vectors \( u = (1, 0, 0), v = (0, 1, 0), w = (0, 0, 1) \) are three linearly independent zeros of \( f(x) \) and \( Z = S(u) \cup S(v, w) \). The condition \( u^T A u < 0 \) for all \( u \in Z \) can easily be verified.

References