ABSTRACT. Let $K_n$ denote the set of all $n \times n$ nonnegative matrices with entry sum $n$. For $X \in K_n$ with row sum vector $(r_1, \ldots, r_n)$, column sum vector $(c_1, \ldots, c_n)$, let \( \phi(X) = \prod_i r_i + \prod_j c_j - \text{per}X \). Dittert's conjecture asserts that $\phi(X) < 2 - n!/n^n$ for all $X \in K_n$ with equality if and only if $X = [1/n]_{n \times n}$. This paper investigates some properties of a certain subclass of $K_n$ related to the function $\phi$ and the Dittert's conjecture.

KEY WORDS AND PHRASES. Permanent, Dittert's function, $A$-admissible matrix.

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1. INTRODUCTION.

Let $K_n$ denote the set of all $n \times n$ nonnegative matrices whose entries have sum $n$, and let $\phi$ denote a real valued function of $K_n$ defined by

\[
\phi(X) = \prod_{i=1}^{n} \sum_{j=1}^{n} x_{ij} + \prod_{j=1}^{n} \sum_{i=1}^{n} x_{ij} - \text{per}X
\]

for $X = [x_{ij}] \in K_n$ where $\text{per}X$ stands for the permanent of $X$;

\[
\text{per}X = \sum_{\sigma \in S_n} x_{\sigma(1)} \cdots x_{\sigma(n)}.
\]

Let $J_n$ denote the $n \times n$ matrix all of whose entries are $1/n$. For the function $\phi$ there is a conjecture due to Eric Dittert.

CONJECTURE (Marcus and Merris [1], Conjecture 28). For $A \in K_n$,

\[
\phi(A) < 2 - \frac{n!}{n^n}
\]

with equality if and only if $A = J_n$. 

In this paper, we will call \( \phi \) the Dittert's function. It is proved that the Dittert's conjecture is true for \( n \leq 3 \) (Marcus and Merris [1], Sinkhorn [2], and Hwang [3]). For a matrix \( X \in K_n \) whose row sum vector is \((r_1, \ldots, r_n)\) and whose column sum vector is \((c_1, \ldots, c_n)\), let

\[
\bar{r}_i = r_1 \cdots r_{i-1} r_{i+1} \cdots r_n \quad (i=1, \ldots, n),
\]

\[
\bar{c}_j = c_1 \cdots c_{j-1} c_{j+1} \cdots c_n \quad (j=1, \ldots, n)
\]

and

\[
\phi_{ij}(X) = \bar{r}_j + \bar{c}_j - \text{per}(X)_{ij} \quad (i,j = 1, 2, \ldots, n)
\]

where \( X(i|j) \) denotes the matrix obtained from \( X \) by deleting the row \( i \) and column \( j \). A matrix \( A \in K_n \) is called a \( \phi \)-maximizing matrix on \( K_n \) if \( \phi(A) > \phi(X) \) for all \( X \in K_n \). In [3], the following results are proved.

**THEOREM A.** If \( A = [a_{ij}] \) is a \( \phi \)-maximizing matrix on \( K_n \), then

\[
\phi_{ij}(A) = \begin{cases} 
\phi(A) & \text{if } a_{ij} > 0 \\
\phi(A) - a_{ij} & \text{if } a_{ij} = 0.
\end{cases}
\]

**THEOREM B.** If, for every \( \phi \)-maximizing matrix \( A \) on \( K_n \), \( \phi_{ij}(A) = \phi(A) \) for all \( i,j = 1, \ldots, n \), then \( J_n \) is the unique \( \phi \)-maximizing matrix on \( K_n \).

We see that \( \phi(A) > 0 \) for all \( A \in K_n \). For \( A \in K_n \) with row sum vector \((r_1, \ldots, r_n)\) and column sum vector \((c_1, \ldots, c_n)\), if either \( r_1 \cdots r_n > 0 \) or \( c_1 \cdots c_n > 0 \), then \( \phi(A) > 0 \). Now, for \( A \in K_n \) with \( \phi(A) > 0 \), let \( A^* = [a^*_{ij}] \) denote the \( n \times n \) matrix defined by

\[
a^*_{ij} = \frac{\phi_{ij}(A)}{\phi(A)} \quad (i,j = 1, \ldots, n).
\]

For \( A \in K_n \), we say that \( A \in K_n \) with \( \phi(A) > 0 \) is \( A \)-admissible (or \( A \) is admissible by \( A \)) if \( \text{tr}(A^T A)^* > n \) where \( A^T \) denotes the transpose of \( A \) and \( \text{tr} \) denotes the trace function. Let \( \mathcal{C}(A) \) denotes the set of all \( A \)-admissible matrices.

It follows from Theorem A that every \( \phi \)-maximizing matrix \( A \) is self-admissible i.e. \( A \in \mathcal{C}(A) \).

If for each \( \phi \)-maximizing matrix \( A \) there exists a positive matrix \( \Lambda \in K_n \) such that \( A \in \mathcal{C}(\Lambda) \), then the Dittert's conjecture is true (See section 2).

In such a point of view, it would be interesting to study the classes \( (\Lambda) \) for some particular matrices \( \Lambda \in K_n \). Such a matrix \( \Lambda \) should be one which is most likely to possess the property that all \( \phi \)-maximizing matrices on \( K_n \) are \( \Lambda \)-admissible.

In this paper we find some matrices in \( \mathcal{C}(\Lambda) \) for certain \( \Lambda \)'s and investigate some properties of the Dittert's function related to the class \( \mathcal{C}(\Lambda) \).

2. **THE CLASS \( \mathcal{C}(\Lambda) \) AND \( \phi \)-MAXIMIZING MATRICES.**

From now on let \( \text{Max}(K_n) \) denote the set of all \( \phi \)-maximizing matrices on \( K_n \).
THEOREM 2.1. If each $A \in \text{Max}(K_n)$ is admissible by a positive matrix in $K_n$, then $\text{Max}(K_n) = \{ J_n \}$, i.e., the Dittrert's conjecture holds.

PROOF. Let $A \in \text{Max}(K_n)$ and let $A = [a_{ij}] \in K_n$ be a positive matrix such that $A \in C(A)$. Then

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{ij} \frac{\phi_{ij}(A)}{\phi(A)} - \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{ij} = n$$

by Theorem A. Therefore the inequalities in (2.1) are all equalities and hence $\phi_{ij}(A) = \phi(A)$ for all $i, j = 1, 2, \ldots, n$ since $A$ is a positive matrix. Now the assertion of the theorem follows from Theorem B.

For $A \in K_n$ with row sum vector $(r_1, \ldots, r_n)$ and column sum vector $(c_1, \ldots, c_n)$, let $A = [a_{ij}]$ denote the $n \times n$ matrix defined by

$$a_{ij} = \frac{r_i c_j}{n} \quad (i, j = 1, \ldots, n).$$

Since $\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} = n^2$ we see that $A \in K_n$. In particular if $A \in \text{Max}(K_n)$, then $A$ is a positive matrix since $r_i > 0$, $c_j > 0$ for all $i, j = 1, \ldots, n$ because $\text{per} A > 0$ [2].

We believe that every $A \in \text{Max}(K_n)$ is $\hat{A}$-admissible, which we can not prove yet. We may ask which matrices $A \in K_n$ are $\hat{A}$-admissible and which are not. We have an answer to this question.

THEOREM 2.2. If $A$ is positive semidefinite symmetric matrix in $K_n$, then $A$ is $\hat{A}$-admissible.

PROOF. Let $A$ be a p.s.d. symmetric matrix in $K_n$ and let $r_i$ be the $i$-th row sum of $A(i=1, \ldots, n)$. Then the condition that $A$ is $\hat{A}$-admissible is equivalent to

$$\sum_{i=1}^{n} \sum_{j=1}^{n} r_i r_j \phi_{ij}(A) > n^2 \phi(A).$$

Let $r=r_1 \ldots r_n$ and let $\bar{r}_i = r_1 \ldots r_{i-1} r_{i+1} \ldots r_n \quad (i=1, \ldots, n)$. Then

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{r_i r_j}{r} \phi_{ij}(A) = \sum_{i=1}^{n} \sum_{j=1}^{n} r_i r_j (\bar{r}_i + \bar{r}_j - \text{per} A(i|j))$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} [(r_i + r_j) r - r_i r_j \text{per} A(i|j)]$$

$$= 2n^2 r - \sum_{i=1}^{n} \sum_{j=1}^{n} r_i r_j \text{per} A(i|j).$$

Since

$$\sum_{i=1}^{n} \sum_{j=1}^{n} r_i r_j \text{per} A(i|j) < n^2 \text{per} A$$

by a theorem of Marcus and Merris [4], we have
and the proof is complete.

Note that not every matrix $A \in K_n$ is $A$-admissible. For $n=2$, the matrix

$$A_x = \begin{bmatrix} 2-2x & x \\ x & 0 \end{bmatrix}$$

in $K_2$ is not $A_x$-admissible if $0 < x < \frac{1}{2}$. For $n > 3$, we have an

EXAMPLE 2.1. Let $T_n$ denote the following $n \times n$ matrix.

$$T_n = \begin{bmatrix} 0 & \cdots & \frac{1}{n-1} \\ \vdots & \ddots & \vdots \\ \frac{1}{n-1} & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix}$$

Then $T_n \in K_n$ and $(r_1, \ldots, r_n) = (1, \ldots, 1), (c_1, \ldots, c_n) = (2, \frac{n-2}{n-1}, \ldots, \frac{n-2}{n-1})$. We have

$$n^2 \phi(T_n) - \sum_{i=1}^{n-1} \sum_{j=1}^{n} r_i c_{ij} \phi_{ij}(T_n) = 2 \frac{(n-1)!}{(n-1)^{n-2}} > 0$$

so that $T_n \in J_n(T_n)$ and hence that $T_n$ is not $T_n$-admissible.

3. THE CLASS $J_n$ AND THE MONOTONICITY OF THE DITTERN'S FUNCTION.

Another candidate for positive $A \in K_n$ with "good" $\phi(A)$ is the matrix $J_n$. A nonnegative square matrix is called a doubly stochastic matrix if all the row sums and column sums are equal to 1. It is conjectured that every $n \times n$ doubly stochastic matrix is $J_n$-admissible (Dokovic [5] and Minc [6]) but this still remains open. Here we have to notice that $A$ is $J_n$-admissible (i.e. $A \in J_n$) if and only if

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \phi_{ij}(A) > n^2 \phi(A).$$

We can show that $\zeta(J_n) \notin K_n$ for $n > 3$ (see Example 3.1). However it seems that $\text{Max}(K_n) \neq J_n$. It is clear that $J_n$ and the $n \times n$ identity matrix $I_n$ are $J_n$-admissible. We can show that all diagonal matrices in $K_n$ are also $J_n$-admissible.

THEOREM 3.1. Every diagonal matrix in $K_n$ is $J_n$-admissible.

PROOF. Let $A = \text{diag}(a_1, \ldots, a_n) \in K_n$, $a = a_1, \ldots, a_n$ and $\overline{a} = \overline{a_1}, \ldots, \overline{a_n-1}$ $a_1+\ldots+a_n$ $(i=1, \ldots, n)$. If $a=0$, there is nothing to prove. Suppose $a > 0$. Then

$$\phi(A) = a$$

and

$$\frac{1}{n} \sum_{i=1}^{n} \phi_{ij}(A) = \frac{1}{n} \sum_{i=1}^{n} (\overline{a_i} + \overline{a_j}) - \frac{1}{n} \overline{a_i} \overline{a_j}.$$
\[ \phi(A) < \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \phi_{ij}(A) \]

if \( n > 2 \), and the proof is complete.

The Dittert's function \( \phi \) has some nice behavior on the set \( \overline{\phi}(\mathbb{J}_n) \) namely that \( \phi \) is monotone on the straight line segment joining \( \mathbb{J}_n \) and \( A \in \overline{\phi}(\mathbb{J}_n) \) whenever the line segment lies in \( \overline{\phi}(\mathbb{J}_n) \). To show this, let \( \Delta \) be a function define by

\[ \Delta(X) = \phi(X) - \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \phi_{ij}(X), \quad X \in \mathbb{K}_n. \]

Let \( A = [a_{ij}] \in \mathbb{K}_n \) have row sum vector \( (r_1, \ldots, r_n) \) and column sum vector \( (c_1, \ldots, c_n) \). For a real number \( t, 0 < t < 1 \), let \( A_t = (1-t)\mathbb{J}_n + tA = [a_{ij}(t)] \) and let the row sum vector and the column sum vector of \( A_t \) be \( (r_1(t), \ldots, r_n(t)) \) and \( (c_1(t), \ldots, c_n(t)) \) respectively.

Letting

\[ r(t) = r_1(t) \ldots r_n(t), \]
\[ c(t) = c_1(t) \ldots c_n(t), \]
\[ \bar{r}_i(t) = r_1(t) \ldots r_{i-1}(t)r_{i+1}(t) \ldots r_n(t), \quad (i=1, \ldots, n), \]
\[ \bar{c}_j(t) = c_1(t) \ldots c_{j-1}(t)c_{j+1}(t) \ldots c_n(t), \quad (j=1, \ldots, n), \]

we compute, for \( t > 0 \), that

\[ \frac{d}{dt} r(t) = \frac{1}{t} \sum_{i=1}^{n} (r(t) - \bar{r}_i(t)) \]
\[ = \frac{n}{t} \{ r(t) - \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \bar{r}_i(t) \}, \]
\[ \frac{d}{dt} c(t) = \frac{1}{t} \sum_{j=1}^{n} (c(t) - \bar{c}_j(t)) \]
\[ = \frac{n}{t} \{ c(t) - \frac{1}{n^2} \sum_{j=1}^{n} \bar{c}_j(t) \}, \]
\[ \frac{d}{dt} \text{per}A_t = \frac{n}{t} \{ \text{per}A_t - \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \text{per}A_t(i,j) \} \]

so that

\[ \frac{d}{dt} \phi(A_t) = \frac{n}{t} \{ r(t) + c(t) - \text{per}A_t \]
\[ - \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} [\bar{r}_i(t) + \bar{c}_j(t) - \text{per}A_t(i,j)] \} \]
which is
\[
\frac{d}{dt} \phi(A_t) = \frac{n}{t} \Delta(A_t).
\]
Thus we have the following

**Theorem 3.2.** Let \( A \in K_n \). If \( A_t \in C(J_n) \) for all \( t, 0 < t < 1 \), then the Dirrert's function is monotone decreasing on the straight line segment from \( J_n \) to \( A \).

It is not hard to show that, for any \( A \in K_2 \),
\[
\frac{1}{2^2} \sum_{i=1}^2 \sum_{j=1}^2 \phi_{ij}(A) = \frac{3}{2}.
\]

On the other hand, the validity of Dittert's conjecture for \( n=2 \) gives us that
\[
\frac{3}{2} = \phi(J_n) \geq \phi(A).
\]

Therefore it follows that \( K_2 = C(J_2) \). However it does not hold in general that \( K_n = C(J_n) \).

**Example 3.1.** Let
\[
U_n = \begin{bmatrix}
\frac{n}{n+1} & \frac{n}{n+1} \\
\frac{n}{n+1} & 0 & 0 \\
\frac{n}{n+1} & 0 & \ddots \\
& \ddots & \ddots & \ddots \\
& & 0 & \frac{n}{n+1} & 0 \\
\end{bmatrix}
\]
and let
\[
U_3 = \begin{bmatrix}
0 & 3/4 & 3/4 \\
3/4 & 0 & 0 \\
3/4 & 0 & 0 \\
\end{bmatrix}
\]

Then
\[
\phi(U_n) = 4 \left( \frac{n}{n+1} \right)^n
\]
and
Hence

\[\sum_{i=1}^{n} \sum_{j=1}^{n} \phi_{ij}(U_n) = \phi(U_n) \times (\text{sum of entries of } U_n^*)\]

\[= 4\left(\frac{n}{n+1}\right)^{n-1} \left(2 + 4(n-3)^2 + 3(6n-10)\right)\]

\[= \left(\frac{n}{n+1}\right)^{n-1} (4n^2 - 6n + 8).\]

Thus we have

\[n^2 \phi(U_n) - \sum_{i=1}^{n} \sum_{j=1}^{n} \phi_{ij}(U_n) = \left(\frac{n}{n+1}\right)^{n-1} \left(\frac{4n^3}{n+1} - 4n^2 + 6n - 8\right)\]

\[= \frac{n}{(n+1)^{n-1}} (2n^2 - 2n - 8),\]

which is positive for all \(n > 3\), telling us that \(U_n\) is not \(J_n\)-admissible.

4. CONCLUDING REMARKS.

If, for every \(A \in \text{Max}(K_n)\), we could find a positive matrix \(A \in K_n\) such that \(A\) is admissible by \(A\), it would prove the Dittert's conjecture by Theorem 2.1. It seems to us that the matrices \(A\) or \(J_n\) are two of the strongest candidates for such matrices. However we may not expect to have a positive matrix \(A \in K_n\) such that all the matrices in \(K_n\) are \(A\)-admissible.

We shall close our discussion here by giving some further research problems.

PROBLEM 4.1. Determine whether there exists a positive matrix \(A \in K_n\) admitting all matrices in \(K_n\).

We conjecture that such a matrix does not exist.

It is proved that every p.s.d. symmetric doubly stochastic matrix is \(J_n\)-admissible [4], from which it follows that the permanent function is monotone increasing on the straight line segment from \(J_n\) to any p.s.d. symmetric doubly stochastic matrix (Hwang [7]).

PROBLEM 4.1. Determine whether every p.s.d. symmetric matrix in \(K_n\) is \(J_n\)-admissible.
If every p.s.d. symmetric matrix in $K_n$ is $J_n$-admissible, then it follows from Theorem 3.2 that the Dittert's function is monotone decreasing on the straight line segment from $J_n$ to any p.s.d. symmetric matrix in $K_n$. We conjecture that the Problem 4.1 will have an affirmative answer.

**PROBLEM 4.3.** Is every $\phi$-maximizing matrix $A$ on $K_n$ $A$-admissible or $J_n$-admissible?

If Problem 4.3 has an affirmative answer, it would prove the Dittert's conjecture as we stated earlier.

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