Remarks on a Fixed-Point Theorem of Gerald Jungck

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Abstract. Jungck [1] obtained a fixed-point theorem for a pair of continuous self-mappings on a complete metric space. Recently, Barada K. Ray [2] extended the theorem of Jungck [1] for three self-mappings on a complete metric space. In the present paper we omit the continuity of the mapping used by Ray [2] and replace his four conditions by a single condition. Our results so obtained generalize and/or unify fixed-point theorems of Jungck [1], Ray [2], Rhoades [3], Cirić [4], Pal and Maiti [5], and Sharma and Yuel [6].

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1. Introduction.

We quote two theorems:

Theorem 1. (Jungck [1]). If $S$ and $T$ are continuous mappings of a complete metric space $(X,d)$ into itself such that

i) $S(X) \subseteq T(X)$,

ii) $ST = TS$, and

iii) $d(Sx, Sy) < \alpha d(Tx, Ty)$ for every pair of points $x, y \in X$ and for $\alpha \in (0,1)$, then

$F_S = F_T = F_{S \cup T} = \{u\}$ for some $u \in X$,

where $F_S = \{x \in X: x = Sx\}$, $F_T = \{x \in X: x = Tx\}$

and $F_{S \cup T} = \{x \in X: x = Sx = Tx\}$.

Theorem 2 (Ray [2]). Let $T$ be a continuous mapping and $T_1$ and $T_2$ be any other two mappings of a complete metric space $(X,d)$ into itself such that

i) $T_{i+1} = T_i T_i, i = 1, 2$,

ii) $U^\infty T_i(X) \subseteq T(X)$, and

iii) at least one of the following is satisfied for every pair of points $x, y \in X$: 
\[ d(T_1x, T_2y) < \frac{\alpha d(T_1y, T_2y) + d(T_1x, T_2x)}{1 + d(Tx, Ty)} + \beta d(Tx, Ty), \]

where \( 0 < \alpha, \beta, \alpha + \beta < 1, \) \( (1.1) \)

\[ d(T_1x, T_2y) < \lambda \max \{ d(Tx, Ty), \frac{1}{2}[d(T_1x, T_2y) + d(Ty, T_1y)] \} \]

where \( 0 < \lambda < 1, \) \( (1.2) \)

\[ d(T_1x, T_2y) < \mu \max \{ d(Tx, Ty), d(T_1x, T_2y), d(Ty, T_1y), d(Tx, T_2y), d(T_1x, T_2y), d(Ty, T_1x) \} \]

where \( 0 < \mu < 1/2, \) \( (1.3) \)

\[ d(T_1x, T_2y) < \max \{ |K_1 d(Tx, Ty) - K_2 d(T_1x, T_2y)|, |K_1 d(Tx, Ty) - K_2 d(Ty, T_1y)| \} \]

where \(-1 < K_2 < K_1 < K_2 + 1 < 2, K_1 < 1.\) \( (1.4) \)

Then \( F_{T_1, T_1, T_2} \) is non-empty, where
\[ F_{T_1, T_1, T_2} = \{ x \in X: x = T_1x = T_2x \} \]

Furthermore, \( F_{T_1} = F_{T_2} = F_{T_1, T_1, T_2} = \{ u \}, \) for some \( u \) in \( X. \)

2. MAIN RESULTS.

Now we give our result.

THEOREM 2.1. Let \((X, d)\) be a complete metric space. Let \( T, T_1, T_2: X \to X \) satisfy (i), (ii) of Theorem 2 and (i) let the following conditions hold for every pair of points \( x, y \) in \( X: \)

\[ d(T_1x, T_2y) < \mu \max \{ d(x, T_1x), d(y, T_2y), d(y, T_1x), d(y, T_2y), \]

\[ \frac{a[l+d(y, T_2y)]d(x, T_1x)}{1 + d(x, y)} \]

\[ + \beta [d(x, T_1x) + d(y, T_2y)] + \nu [d(y, T_1x) + d(x, T_2y)] \]

\[ + \delta d(x, y) \}, \]

\[ |K_1 d(x, y) - K_2 d(y, T_1x)|, \]

\[ |K_1 d(x, y) - K_2 d(y, T_2y)| \]
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where \(0 < \mu < 1, \alpha, \beta, \nu, \delta > 0, \alpha + \beta + \nu + \delta < 1, 2\nu + \delta < 1,\)

\[
0 < \frac{\mu(\beta + \nu + \delta)}{1 - \mu(\alpha + \beta + \nu)} < 1, \quad -1 < K_2 < K_1 < 1 + \mu K_2 < 2, \quad K_1 < 1.
\]

Then \(F_{T_1, T_2}\) is non-empty, where

\[
F_{T_1, T_2} = \{x \in X: x = T_1x = T_2x\}
\]

Furthermore, \(F_{T_1, T_2} = F_{T_2, T_1} = \{u\}\), for some \(u\) in \(X\).

**PROOF.** Let \(x_0 \in X\), define

\[
x_{2n+1} = T_1x_{2n}, \quad n = 0, 1, 2, \ldots
\]

\[
x_{2n} = T_2x_{2n-1}, \quad n = 1, 2, 3, \ldots
\]

Then, using Theorem 2.1, (i), we have

\[
d(x_{2n+1}, x_{2n}) < Kd(x_{2n}, x_{2n-1})
\]

where \(K = \max \{\mu, \frac{\mu(\beta + \nu + \delta)}{1 - \mu(\alpha + \beta + \nu)}\}r\)

\[
\mu \max \{K_1, K_2, \frac{K_1}{1 + \mu K_2}\}, \quad K_1 > 0,
\]

\[
u \max \{K_1, K_2, \frac{K_1}{1 - \mu K_2}\}, \quad K_1 < 0.
\]

\(\{x_n\}\) is a Cauchy sequence. Since \(X\) is complete there exists \(u \in X\) such that \(x_n \to u\) as \(n \to \infty\).

Now,

\[
d(T_1Tu, x_{2n}) = d(T_1Tu, T_2Tx_{2n-1}).
\]

Then using Theorem 2.1 (i) and allowing \(n \to \infty\) such that \(x_{2n} \to u, x_{2n-1} \to u\) etc, we have \(u = T_1Tu\). Hence \(u = T_1Tu = TTu\) using Theorem 2 (i). Further,

\[
d(x_{2n+1}, T_2Tu) = d(T_1Tx_{2n}, T_2Tu).
\]

Again using Theorem 2 (i) and allowing \(n \to \infty\) such that \(x_{2n} \to u, x_{2n-1} \to u\) etc, we have \(u = T_2Tu\). Hence \(u = T_2Tu = TTu\).

Now, let \(v\) denote any common fixed point of \(T_1T\) and \(T_2T\). From Theorem 2.1 (i), it is easy to see that \(u = v\) since \(2\nu + \delta < 1\). For proving \(u = Tu\) we have

\[
d(Tu, u) = d(TT_1Tu, T_2Tu) = d(T_1Tu, T_2Tu)
\]

which yields \(Tu = u\) using Theorem 2.1 (i). Hence \(u = T_1Tu = T_2u\). Similarly, \(u = T_2Tu = T_2u\). Hence, \(u = Tu = T_1u = T_2u\) which shows that \(F_{T_1, T_2}\) are non-empty. Then we
can see that \( F_{T_1} \circ F_{T_2} = F_{T_1 T_2} = \{u\} \) for some \( u \) in \( X \). This completes the proof.

EXAMPLE. Let \( X = [0,1] \) with Euclidean metric \( d \). Let \( T_1 x = x, 0 < x < 1, T_1 x = \frac{1}{2}, x = 1 \), \( T_2 x = x, 0 < x < 1, T_2 x = \frac{x}{4}, x = 1 \). Here \( T_1, T_2 \) are all discontinuous at \( x = 1 \) and have a unique common fixed point \( x = 0 \). Take \( x = \frac{1}{2}, y = \frac{1}{4} \). Obviously all the conditions (i), (ii) of Theorem 2 and (i) of Theorem 2.1 hold true. Hence the result.

REMARKS. (1) Contractive Definition 20 of Rhoades [3] is a special case of condition (i) of Theorem 2.1. (2) Theorem 1 of Ciric [4], Theorem 1 of Pal and Maiti [5], and Theorem 4 of Sharma and Yuel [6] are special cases of Theorem 2.1.

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REFERENCES