A COMMUTATIVITY THEOREM FOR LEFT $s$-UNITAL RINGS

HAMZA A.S. ABUJABAL

Department of Mathematics
Faculty of Science
King Abdul-Aziz University
P.O. BOX 9028, Jeddah - 21413
Saudi Arabia

(Received June 2, 1989 and in revised form July 25, 1989)

ABSTRACT. In this paper we generalize some well-known commutativity theorems for associative rings as follows: Let $R$ be a left $s$-unital ring. If there exist non-negative integers $m > 1$, $k > 0$, and $n > 0$ such that for any $x, y$ in $R$, $[x - x^ny^m, x] = 0$, then $R$ is commutative.

KEY WORDS AND PHRASES. Associative ring, $s$-unital ring, ring with unity, commutativity of rings.

1980 AMS SUBJECT CLASSIFICATION CODE. 16A70

1. INTRODUCTION.

Throughout this paper, $R$ denotes an associative ring (may be without unity), $Z(R)$ represents the center of $R$, $N$ the set of all nilpotent elements of $R$, $N'$ the set of all zero divisors of $R$, and $C(R)$ the commutator ideal of $R$. For any $x, y \in R$, we write $[x, y] = xy - yx$.

As stated in Hirano and Kobayashi [1] and Quadri and Khan [2], a ring $R$ is called left (resp. right) $s$-unital if $x \in Rx$ (resp. $x \in xR$) for each $x \in R$. Further, $R$ is called $s$-unital if it is both left as well as right $s$-unital, that is $x \in Rx \cap xR$, for every $x \in R$. If $R$ is $s$-unital (resp. left or right $s$-unital), then for any finite subset $F$ of $R$, there exists an element $e \in R$ such that $ex = e = xe$ (resp. $ex = x$ or $xe = x$) for all $x \in F$. Such an element $e$ will be called a pseudo-identity (resp. pseudo left identity or pseudo right identity) of $F$ in $R$.

The famous Jacobson theorem stated that any ring $R$ in which for every $x \in R$ there exists a positive integer $n = n(x) > 1$ such that $x^n = x$ is commutative, has been generalized as follows: if for each pair $x, y \in R$ there exists a positive integer $n = n(x, y) > 1$ such that $(xy)^n = xy$, then $R$ is commutative. Recently, Ashraf and Quadri [3] investigated the commutativity of the rings satisfying the following condition: For all $x, y \in R$ there is a fixed integer $n > 1$ such that $x^ny^n = xy$. In fact, Ashraf and Quadri [3] have generalized the above results as follows: Let $R$ be a ring with unity $1$ in which $[xy - x^ny^m, x] = 0$, for all $x, y \in R$ and fixed integers $m > 1, n > 1$. Then $R$ is commutative.

The objective of this paper is to generalize the above mentioned results. Indeed, we prove the following:
THEOREM 1.1. Let R be a left s-unital ring with the property that

\( (P) \) "there exist positive integers \( m > 1, k > 0, \) and \( n > 0 \)

such that \( x^k y - x^n y^m, x = 0 \) for all \( x, y \in R \)."

Then R is commutative.

We notice that the property (P) of the above theorem can be rewritten as follows:

\[ x^k [x, y] = x^n [x, y^m]. \] (1.1)

Thus for any integer \( t \geq 1 \), we have

\[ x^{tk} [x, y] = x^{(t-1)k} (x^k [x, y]) \]
\[ = x^{(t-1)k} (x^n [x, y^m]) \]
\[ = x^{(t-2)k} (x^{n-1} x [x, y^m]) \]
\[ = x^{(t-2)k} (x^{2n-1} [x, y^m]) \]
\[ = \cdots \]

By repeating the above process and using (1.1), we get

\[ x^{tk} [x, y] = x^{tn} [x, y^m]. \] (1.2)

2. PRELIMINARY LEMMAS.

In preparation for the proof of the above theorem we start by stating without proof the following well-known Lemmas.

LEMMA 2.1 (Bell [4, Lemma 1]). Suppose \( x \) and \( y \) are elements of a ring \( R \) with unity 1, satisfying \( x^m y = 0 \) and \( (1+x)^m y = 0 \) for some positive integer \( m \). Then \( y = 0 \).

LEMMA 2.2. (Bell [5, Lemma 3]). Let \( x \) and \( y \) be in \( R \). If \( [x,y] \) commutes with \( x \), then \( [x^k, y] = k x^{k-1} [x, y] \) for all positive integers \( k \).

LEMMA 2.3 ([2, Lemma 3]). Let \( R \) be a ring with unity 1. If \( (1 - y^k)x = 0 \), then \( (1 - y^{km}) x = 0 \), for any positive integers \( m \) and \( k \).

LEMMA 2.4 ([1, Proposition 2]). Let \( f \) be a polynomial in non-commuting indeterminates \( x_1, x_2, \ldots, x_n \) with integer coefficients. Then the following statements are equivalent:

1) For any ring \( R \) satisfying \( f = 0 \), \( C(R) \) is a nil ideal.
2) Every semi-prime ring satisfying \( f = 0 \) is commutative.
3) For every prime \( p \), \( (\mathbb{GF}(p))_2 \) fails to satisfy \( f = 0 \).

3. MAIN RESULTS.

The following lemmas will be used in the proof our main theorem.

LEMMA 3.1. Let \( R \) be a left \( s \)-unital ring satisfying \( x^k y - x^n y^m, x = 0 \), for each \( x, y \in R \) and any non-negative integers \( k, n, m \) and \( m > 1 \). Then \( R \) is \( s \)-unital.

PROOF. Let \( u \in \mathbb{N} \). Then for any \( x \in R \), and \( t > 1 \), we have \( x^{tk} [x, u] = x^{tn} [x, u^m] \).
For sufficiently large $t$, we have $x^{tk}[x,u] = x^{tn}[x,u^m]^t = 0$, since $u$ is nilpotent and $u^m = 0$.

Since $R$ is a left $s$-unital ring, we have $u = cu$ for some $c \in R$. But $e^{tk}[e,u] = 0$ which gives $u = cu$. For arbitrary $x \in R$, there exists $e' \in R$ such that $e'x = x$. Further, for some $e'' \in R$, we have $e''e' = e'$. Thus $e''e = e$ and $(x - xe'')e'' = 0$, that is $(x - xe'') \in N$. Since $e''(x - xe'')e' = x - xe''$, we have $x - xe'' = (x - xe')e'' = 0$ which implies $x = xe''$. Hence $R$ is $s$-unital.

**Lemma 3.2.** Let $R$ be a ring with unity $1$ which satisfies the property $(P)$. Then every nilpotent element of $R$ is central.

**Proof.** Let $u$ be a nilpotent element of $R$. Then by (1.2) for any $x \in R$ and a positive integer $t > 1$ we have $x^{tk}[x,u] = x^{tn}[x,u^m]^t$. But $u \in N$, then $u^m = 0$, for sufficiently large $t$, and hence $x^{tk}[x,u] = 0$ for each $x \in R$. By Lemma 2.1 this yields $[x,u] = 0$, which forces $N \subseteq Z(R)$. Thus every nilpotent element of $R$ is central.

**Lemma 3.3.** Let $R$ be a ring with unity $1$ which satisfies the property $(P)$, then $C(R) \subseteq Z(R)$.

**Proof.** Now, $R$ satisfies $[x^k y - x^n y^m, x] = 0$ for all $x, y \in R$, which is a polynomial identity with relatively prime integral coefficients. Let $x = e_{12} = (0 1)$ and $y = e_{21} = (1 0)$, we find that no ring of $2 \times 2$ matrices over $GF(p)$, $p$ a prime, satisfies the above polynomial identity. Hence by Lemma 2.4, the commutator ideal $C(R)$ of $R$ is nil. Therefore $C(R) \subseteq Z(R)$.

In view of Lemma 3.3 it is guaranteed that the conclusion of Lemma 2.2 holds for each pair of elements $x, y$ in a ring $R$ with unity $1$ which satisfies the property $(P)$.

**Lemma 3.4.** Let $R$ be a ring with unity $1$, satisfying $(P)$, then $R$ is commutative.

**Proof.** Since $R$ is isomorphic to a subdirect sum of subdirectly irreducible rings $R_i$ each of which as a homomorphic image of $R$ satisfies the property $(P)$ placed on $R$, $R$ itself can be assumed to be a subdirectly irreducible ring. Let $S$ be the intersection of all its non-zero ideals, then $S \neq (0)$.

Let $k = n = 0$, in (1.1). Then we have $[x,y] = [x,y^m]$ or $[x,y - y^m] = 0$ for all $x, y \in R$. This forces commutativity of $R$ by Herstein [6, Theorem 18]. Next, we assume $k = n = 1$ in (1.1). Then replacing $x$ by $(x + 1)$, we obtain $[x,y] = [x,y^m]$, for every $x, y \in R$, and again by [6, Theorem 18] $R$ is commutative. If $(k,n) = (1,0)$, then $x[x,y] = [x,y^m]$ and hence by replacing $x$ by $(x + 1)$ we have $[x,y] = 0$, for all $x, y \in R$. Therefore $R$ is commutative. If $(k,n) = (0,1)$, then $[x,y] = x[x,y^m]$, and hence by replacing $x$ by $(x + 1)$ we have $[x,y] = 0$, for all $x, y \in R$. Thus $[x,y] = x[x,y^m] = 0$ for all $x, y \in R$. Thus $R$ is commutative.

Next, we suppose that $k > 1$, and $n > 1$. Let $q = 2^n - 2$ be a positive integer. Then by (1.1) we have

$$q x^k [x,y] = 2^m x^k [x,y] - 2 x^k [x,y]$$

$$= 2^m x^n [x,y^m] - x^k [x,2y]$$

$$= x^n [x, (2y)^m] - x^n [x, (2y)^m]$$

$$= 0.$$
that is, \( qx^k [x,y] = 0 \). By replacing \( x \) by \((x + 1)\) and using Lemma 2.1, this yields \( q[x,y] = 0 \) for all \( x,y \in R \). Now combining Lemma 3.3 with Lemma 2.2, we get

\[
[x^q,y] = q x^{q-1} [x,y] = 0
\]

which yields

\[
x^q \in Z(R) \quad \text{for all } x,y \in R.
\] (3.1)

Replacing \( y \) by \( y^m \) in (1.1), we get

\[
x^k [x,y^m] = x^n [x,(y^m)^m].
\] (3.2)

By applying Lemma 3.3 and Lemma 2.2, we obtain

\[
x^k [x,y^m] = [x,y^m] x^k
\]

\[
= my^{m-1} [x,y] x^k
\]

\[
= my^{m-1} y [x,y]
\]

\[
= m y^{m-1} x^n [x,y^m]
\]

and, using similar techniques, we get

\[
x^n [x, (y^m)^m] = [x, (y^m)^m] x^n
\]

\[
= m(y^m)^{m-1} [x,y^m] x^n
\]

\[
= m y^{m-2} [x,y^m] x^n
\]

\[
= m y^{m-1} y^{(m-1)^2} [x,y^m] x^n.
\]

Thus (3.2) gives

\[
m y^{m-1} (1 - y^{(m-1)^2}) [x,y^m] x^n = 0.
\] (3.3)

Again the usual argument of replacing \( x \) by \((x + 1)\) in (3.3) and applying Lemma 2.1 yields \( m y^{m-1} (1 - y^{(m-1)^2}) [x,y^m] = 0 \). Then by Lemma 3.3 and Lemma 2.3 we have

\[
m y^{(m-1)} (1 - y^{(m-1)^2}) [x,y^m] = 0.
\] (3.4)

Next, we claim that \( N' \subseteq Z(R) \). Let \( a \in N' \), then by (3.1) \( a^{q(m-1)^2} \in N' \cap Z(R) \), and \( S a^{q(m-1)^2} = 0 \). Since by (3.4), \( m a^{(m-1)} (1 - a^{q(m-1)^2}) [x,a^m] = 0 \), that is, \((1 - a^{q(m-1)^2}) m a^{m-1} [x,a^m] = 0\).
Now, if \( m^{-1}a^m \neq 0 \), then \((1-aq(m-1)^2) \in N'\), and so \( S(1-aq(m-1)^2) = 0 \) which leads to the contradiction that \( S = (0) \). Hence \( m^{-1}a^m = 0 \). From (1.1) and using Lemma 3.2 repeatedly we get

\[
x^{2k}[x,a] = x^k(x^k[x,a^m]) = x^k(x^n[x,a^m]) = x^n(x^k[x,a^m]) = x^{2n}[x,(a^m)^m] = x^{2n}m(a^m)^{-1}[x,a^m] = x^{2n}a^{-1}(m-1)^2[x,a^m] = x^{2n}a^{-2}(m-1)^2m^{-1}[x,a^m] = 0.
\]

This implies that \( x^{2k}[x,a] = 0 \), and so the usual argument of replacing \( x \) by \((x + 1)\) and using Lemma 2.1 gives \([x,a] = 0\), and hence,

\[ N' \subseteq Z(R). \quad (3.5) \]

Now, for any \( x \in R \), \( x^q \) and \( x^{qm} \) are in \( Z(R) \). Then by (1.1) for any \( y \in R \), we have

\[
(x^q - x^{qm})x^k[x,y] = x^q(x^k[x,y]) - x^{qm}(x^k[x,y]) = x^k(x^q[x,y]) - x^{qm}x^n[x,y^m] = x^k[x,x^qy] - x^n[x,(x^qy)^m] = x^k[x,x^qy] - x^k[x,x^{q-1}].
\]

Therefore \((x^q - x^{qm})x^k[x,y] = 0\), and hence

\[(x - x^{qm-q+1})x^{k+q-1}[x,y] = 0. \quad (3.6)\]

If \( R \) is not commutative then by [6, Theorem 18], there exists an element \( x \in R \) such that \((x - x^t) \notin Z(R)\), where \( t = qm - q + 1 \). This also reveals \( x \notin Z(R) \). Thus neither \((x-x^t)\) nor \( x \) is a zero divisor, and so \((x-x^t) x^{k+q-1} \notin N'\). Hence (3.6) forces that \([x,y] = 0\), for all \( x,y \in R \). Thus \( x \notin Z(R) \) which is a contradiction. Hence \( R \) is commutative.

**Proof of the Theorem.** Let \( R \) be a left \( s \)-unital ring satisfying (P), then by Lemma 3.1, \( R \) is \( s \)-unital. Therefore, in view of [1, Proposition 1] and Lemma 3.4, \( R \) is commutative, if \( R \) with 1 satisfying (P) is commutative.

**Corollary 3.1 ([3, Theorem]).** Let \( R \) be a ring with unity 1 in which \( [x^ny^m,x] = 0 \) for all \( x,y \in R \) and fixed integers \( m > 1, n > 1 \). Then \( R \) is commutative.

**Proof.** Actually, \( R \) satisfies the polynomial identity \( x[x,y] = x^n[x,y^m] \) for all \( x,y \in R \) and fixed integers \( m > 1, n > 1 \). Put \( k = 1 \) in (1.1), then \( R \) is commutative by Lemma 3.4.

**Corollary 3.2 (Hirano, Kobayashi, and Tominaga [7, Theorem]).** Let \( m,k \) be fixed non-negative integers. Suppose that \( R \) satisfies the polynomial identity
$x^k [x,y] = [x,y]^m$.

(a) If $R$ is a left s-unital, then $R$ is commutative except when $(m,k) = (1,0)$.

(b) If $R$ is a right s-unital, then $R$ is commutative except when $(m,k) = (1,0)$, and $m = 0$, $k > 0$.

REMARK 3.1. ([7]). In case $k > 0$ and $m = 0$ in Corollary 3.2(b), $R$ need not be commutative. For, let $K$ be a field. Then the non-commutative ring

$R = \left\{ \begin{array}{ll}
0 & \\
1 & \\
0 & \\
1 & \\
\end{array} \right\}$

$a,b \in K$ has a right identity element and satisfies the polynomial identity $x[x,y] = 0$.

ACKNOWLEDGEMENT. I am thankful to Dr. M.S. Khan for his valuable advice.

REFERENCES