A NOTE ON NEIGHBORHOODS OF ANALYTIC FUNCTIONS HAVING POSITIVE REAL PART

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ABSTRACT. Let $P$ denote the set of all functions analytic in the unit disk
$D = \{z \mid |z| < 1\}$ having the form $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$ with $\text{Re}(p(z)) > 0$. For $\delta > 0$, let
$N_\delta(p)$ be those functions $q(z) = 1 + \sum_{k=1}^{\infty} q_k z^k$ analytic in $D$ with $\sum_{k=1}^{\infty} |p_k - q_k| \leq \delta$. We
denote by $P'$ the class of functions analytic in $D$ having the form $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$
with $\text{Re}[zp(z)]' > 0$. We show that $P'$ is a subclass of $P$ and determine $\delta$ so that
$N_\delta(p) = P$ for $p \in P'$.

KEY WORDS AND PHRASES. Functions having positive real part (Carathéodory class),
subordinate function, $\delta$-neighborhood, and convolution (Hadamard product).


I. INTRODUCTION

Let $H$ denote the class of functions $f$ analytic in the unit disk $D = \{z \mid |z| < 1\}$
with $f(0) = 0$ and $f'(0) = 1$. For $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ in $H$ and $\delta > 0$, let the
$\delta$-neighborhood of $f$ be given by $N_\delta(f) = \{g(z) = z + \sum_{k=2}^{\infty} b_k z^k \mid \sum_{k=2}^{\infty} |a_k - b_k| \leq \delta\}$.
For $h(z) = z$, Goodman [1] has shown that $N_1(h) = S^*$ where $S^*$ denotes the class of
univalent functions in $H$ which are starlike with respect to the origin. St. Ruscheweyh [2]
proved that if $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$ lies in $C$, where $C$ denotes the class of convex
univalent functions in $H$, then $N_\delta(f) \subset S^*$ for $\delta_n = \frac{2^{2/n}}{n}$. Fournier [3] found that if
$C$ were replaced by

$$\tilde{f} = \{g \in C \mid |\frac{g''(z)}{g'(z)}| < 1, z \in D\}$$

and $S^*$ by

$$T = \{g \in S^* \mid |\frac{g'(z)}{g(z)} - 1| < 1, z \in D\}$$

then $N_\delta^n(f) \subset T$ for $\delta_n = e^{-1/n}$. Brown [4] extended the results of St. Ruscheweyh and
Fournier and provided simpler proofs. We shall focus on a class of functions directly
related to $S^*$ and to other classes of univalent functions. Let $P$ denote the class of
functions analytic in \( |z| < 1 \) having the form \( p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \) with \( \text{Re}(p(z)) > 0 \) for \( |z| < 1 \). This family is usually called the Carathéodory class. For \( f \in H \), recall that \( f \in S^* \) if and only if \( p(z) = z f'(z)/f(z) \) lies in \( P \).

Let \( P' \) denote the class of functions analytic in \( |z| < 1 \) having the form \( p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \) with \( \text{Re}[zp(z)]' > 0 \) for \( |z| < 1 \). In this paper we shall define a neighborhood of \( p \in P' \) and determine \( \delta > 0 \) so that \( N_\delta(p) \subseteq P' \).

2. PRELIMINARY RESULTS.

We begin by defining \( P \) and \( P' \) in terms of subordination. Recall that \( g \) is subordinate to \( h \), written \( g \prec h \), if \( g(z) = h(w(z)) \) where \( w \) is analytic in \( 1 + z \), \( w(0) = 0 \) and \( |w(z)| < 1 \) for \( |z| < 1 \). Since \( \frac{1+z}{1-z} \) has positive real part in \( |z| < 1 \), is univalent, and is 1 when \( z = 0 \), it is not difficult to show that

\[
p \in P \text{ if and only if } p(z) \prec \frac{1+z}{1-z}
\]

and that

\[
p \in P' \text{ if and only if } [zp(z)]' \prec \frac{1+z}{1-z}.
\]

One can also show that \( P' \subseteq P \). For according to (2.2), if \( p \in P' \) then

\[
[zp(z)]' \prec \frac{1+z}{1-z}
\]

and thus we have

\[
[zp(z)]' \prec \frac{1+z}{1-z}.
\]

Since \( \frac{1+z}{1-z} \) is convex and univalent, we can apply a lemma (see Brown [5], p. 192) to obtain

\[
zp(z) \prec \frac{1+z}{1-z},
\]

from which it follows that

\[
p(z) \prec \frac{1+z}{1-z}.
\]

Hence, by (2.1) \( p \in P \) and \( P' \subseteq P \).

Now let us establish a criterion for a given function to belong to \( P \). By (2.1) \( q \in P \) if and only if \( q(z) \prec \frac{1+z}{1-z} \). Since \( \frac{1+z}{1-z} \) is univalent, then \( q \in P \) if and only if

\[
q(z) \equiv \frac{1 + e^{i\theta}}{1 - e^{i\theta}}, \text{ for } 0 < \theta < 2\pi \text{ and } |z| < 1.
\]

That is,

\[
q \in P \text{ if and only if } (1 - e^{i\theta})q(z) - (1 + e^{i\theta}) \equiv 0,
\]

for \( 0 < \theta < 2\pi \), \( |z| < 1 \).

We can express (2.3) in terms of convolutions. Let \( f \) and \( g \) be analytic in the unit disk \( D \). Recall that if \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) and \( g(z) = \sum_{k=0}^{\infty} b_k z^k \), then the convolution (or Hadamard product) of \( f \) and \( g \), denoted by \( f \ast g \), is

\[
f \ast g = \sum_{k=0}^{\infty} a_k b_k z^k.
\]
Thus, \((1 - e^{i\theta})q(z) - (1 + e^{i\theta})\) can be written as

\[
(1 - e^{i\theta}) \left[ \frac{1}{1-z} * q(z) \right] - (1 + e^{i\theta}) * q(z) = \left( \frac{1 - e^{i\theta}}{1-z} - (1 + e^{i\theta}) \right) * q(z).
\]

Let \(h_\theta(z)\) be defined by

\[
h_\theta(z) = -\frac{1}{2e^{i\theta}} \left[ \frac{1 - e^{i\theta}}{1-z} - (1 + e^{i\theta}) \right].
\]

Then it follows that \(h_\theta(0) = 1\) and for \(0 < \theta < 2\pi, \, |z| < 1, \, q \in P\) if and only if \(h_\theta(z) * q(z) \neq 0\). (2.4)

3. THE MAIN RESULT.

We define a \(\delta\)-neighborhood of \(p\) for \(p \in P\).

DEFINITION. For any \(p(z) = 1 + \sum_k p_k z^k\) in \(P\) and \(\delta > 0\), the \(\delta\)-neighborhood of \(p\), denoted by \(N_\delta(p)\), is

\[
N_\delta(p) = \left\{ q(z) = 1 + \sum_k q_k z^k \mid \sum_k |p_k - q_k| < \delta \right\}.
\]

Our main result is the following theorem.

THEOREM. If \(p(z) = 1 + \sum_k p_k z^k\) belongs to \(P'\), then \(N_\delta(p) \subseteq P\), where \(\delta = 2 \ln 2 - 1 \approx .3862944\). This result is sharp.

We need several lemmas.

LEMMA 1. If \(p \in P'\), then \(z(p*h_\theta)\) is univalent for each \(0 < \theta < 2\pi\).

PROOF. Fix \(0 < \theta < 2\pi\). Then

\[
[z(p*h_\theta)]' = \left[ \frac{-z}{2e^{i\theta}} \left( (1 - e^{i\theta})p(z) - (1 + e^{i\theta}) \right) \right]'.
\]

\[
= -\frac{1}{2} \left[ zp(z) - \frac{1 + e^{i\theta}}{1 - e^{i\theta}} \right] \frac{1 - e^{i\theta}}{e^{i\theta}}
\]

\[
= -\frac{1}{2} \left[ (zp(z))' - \frac{1 + e^{i\theta}}{1 - e^{i\theta}} \right] (1 - e^{i\theta}) e^{-i\theta}. \quad (3.1)
\]

By definition of \(P'\), the range of \((zp(z))'\) for \(|z| < 1\) lies in \(\text{Re}(z) > 0\) and that of \(\frac{1 + e^{i\theta}}{1 - e^{i\theta}}\) lies on the imaginary axis. Thus, we can choose \(\alpha\) so that

\[
\text{Re}(e^{i\alpha}(z(p*h_\theta)(z))') > 0
\]

for \(|z| < 1\), namely \(\alpha = \arg(-(1 - e^{i\theta})^{-1}e^{i\theta})\). By the Noshiro-Warschawski Theorem (Duren [6], p. 47), \(z(p*h_\theta)\) is univalent for each \(\theta, \, 0 < \theta < 2\pi\).

LEMMA 2. If \(p \in P'\), then \(|z(p*h_\theta)|^2 > \frac{1-r}{1+r}\) for \(|z| = r < 1, \, 0 < \theta < 2\pi\).

PROOF. Using expression (3.1) for \(|z(p*h_\theta)|^2\), we define \(F(w) = e^{-i\theta}(1 - e^{i\theta}) (\frac{1 + e^{i\theta}}{1 - e^{i\theta}} - w)\), where \(w = \frac{1 + re^{it}}{1-re^{it}}, \, 0 \leq t \leq 2\pi\). Now \(F(w)\) may be rewritten as

\[
F(w) = e^{-i\theta}(1 + e^{i\theta}) - (1 - e^{i\theta})w, \, 0 < \theta < 2\pi.
\]

Thus,

\[
|F(w)| = |1 + w| \left| \frac{1 - w + e^{i\theta}}{1 + w} \right|
\]
Since \( |1 + w| = \left| 1 + \frac{1 + re^{it}}{1 - re^{it}} \right| = \left| \frac{2}{1 - re^{it}} \right| \geq \frac{2}{1 + r} \), it is clear that 
\[ |F(w)| \geq 2 \frac{1 - r}{1 + r}. \]

Since \( p \in P' \) and (3.1) holds, by letting \( w = [zp(z)]' \) we get the desired inequality. That is
\[ \left| [z(p*h_0)]' \right| \geq \frac{1 - r}{1 + r}. \]

The lemma is proved.

**LEMMA 3.** If \( p \in P' \), then \( |p*h_0| > \delta \), where \( \delta = \int_0^1 \frac{1 - t}{1 + t} dt = 2 \ln 2 - 1. \)

**PROOF.** Let \( p \in P' \). Then by Lemma 1, \( z(p*h_0) \) is univalent. For fixed \( 0 < r < 1 \), choose \( z_0 \) with \( |z_0| = r \) such that
\[ \min |z(p*h_0)| = |z_0(p*h_0)(z_0)|. \]

Since \( z(p*h_0) \) is univalent, the preimage \( L \) of the line segment from 0 to \( z_0(p*h_0)(z_0) \) is an arc inside \( |z| < r \). Hence, for \( |z| < r \) we have
\[ |z(p*h_0)| \geq |z_0(p*h_0)| = \int_0^r \left| [z(p*h_0)]' \right| dz. \]

Accordingly, we apply Lemma 2 to get
\[ \left| [p*h_0](z) \right| \geq \frac{1}{r} \int_0^r \left| [z(p*h_0)]' \right| dz. \]

The function \( g(r) = \frac{2}{r} \ln (1 + r) - 1 \) is decreasing for \( r > 0 \) if \( g'(r) = \frac{-2}{r^2} \ln (1 + r) + \frac{2}{r^2(1 + r)} \). It is not difficult to show that \( r - (1 + r) \ln (1 + r) \leq 0 \) for \( r > 0 \), from which it follows that \( g'(r) < 0 \) for \( r > 0 \). Hence
\[ |p*h_0| \geq 2 \ln 2 - 1. \]

This completes the proof of Lemma 3. Now we may prove the theorem.

**PROOF (OF THEOREM).** Let \( p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \in P' \) and let \( \delta \) be as in Lemma 3. We want to show that every \( q \in N_\delta(p) \) belongs to \( P \), where \( q(z) = 1 + \sum_{k=1}^{\infty} q_k z^k \) is an arbitrary but fixed function in \( N_\delta(p) \). Hence, \( \sum_{k=1}^{\infty} |p_k - q_k| \leq \delta \). Observe that
\[ |h_0*q| = |(h_0*p) + h_0*(q - p)| \]
\[ |h_0(p) - h_0(q - p)| \]
\[ \geq \delta - \frac{1}{2} \sum_{k=1}^{\infty} |q_k - p_k| \]
\[ \geq \delta - \sum_{k=1}^{\infty} (q_k - p_k)z^k > \delta - \delta = 0. \]

Therefore, \( h_0(q) \neq 0 \) for \( |z| < 1 \). By (2.4), it follows that \( q \in F \). Consequently, \( N_0(p) = F \).

Now we prove that the result is sharp. Let \( p(z) \) be defined by \( (zp(z))' = \frac{1+z}{1-z} \).
Then \( p(z) = -1 - \frac{2}{z} \ln (1 - z) \). Now let \( q(z) = p(z) + \delta z = -1 - \frac{2}{z} \ln (1 - z) + \delta z \).
Clearly, \( q \in N_0(p) \). However, as \( z \to -1 \), then \( q(z) \to -1 + 2 \ln 2 - \delta = q(-1) \).
Therefore, if \( \delta > 2 \ln 2 - 1 \), then \( q(-1) < 0 \) and consequently \( \Re q(z) < 0 \) for \( z \) near \(-1\). This contradicts \( \Re q(z) > 0 \) for \( |z| < 1 \). This completes the proof of the theorem.

REFERENCES