HEARING THE SHAPE OF MEMBRANES: FURTHER RESULTS

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ABSTRACT. The spectral function \( \Theta(t) = \sum_{m=1}^{\infty} \exp(-t \lambda_m) \), \( t > 0 \) where \( \{\lambda_m\}_{m=1}^{\infty} \) are the eigenvalues of the Laplacian in \( \mathbb{R}^n \), \( n = 2 \) or 3, is studied for a variety of domains. Particular attention is given to circular and spherical domains with the impedance boundary conditions \( \frac{\partial u}{\partial r} + \gamma_j u = 0 \) on \( \Gamma_j \) (or \( S_j \)), \( j = 1, \ldots, J \) where \( \Gamma_j \) and \( S_j \), \( j = 1, \ldots, J \) are parts of the boundaries of these domains respectively, while \( \gamma_j \), \( j = 1, \ldots, J \) are positive constants.

1. INTRODUCTION.

The underlying problems are to deduce the precise shape of membranes from the complete knowledge of the eigenvalues

\[
0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_m < \ldots + \infty \quad \text{as} \quad m \to \infty,
\]

for the Laplace operator \( \Delta_n \) in \( \mathbb{R}^n \), \( n = 2 \) or 3.

(P1): Let \( \Omega = \{(r, \theta): 0 < r < a, 0 < \theta < 2\pi\} \) be a circular domain of radius \( a \) and boundary \( \Gamma \). Suppose that the eigenvalues (1.1) are given for the eigenvalue equation \( (\Delta^2 + \lambda) u = 0 \) in \( \Omega \) together with the impedance boundary conditions:

\[
(-\frac{\partial}{\partial r} + \gamma_j) u = 0 \quad \text{on} \quad \Gamma_j, \quad j = 1, \ldots, J,
\]

where \( \gamma_j \), \( j = 1, \ldots, J \) are positive constants and the boundary \( \Gamma \) consists of parts \( \Gamma_j \), \( j = 1, \ldots, J \) such that

\[
\Gamma_j = \{(r, \theta): r = a, \alpha_j < \theta < \alpha_{j+1}, \quad j = 1, \ldots, J, \alpha_1 = 0, \alpha_{J+1} = 2\pi\}.
\]

(P2): Let \( \Omega = \{(r, \theta, \phi): 0 < r < a, 0 < \theta < \pi, 0 < \phi < 2\pi\} \) be a spherical domain of radius \( a \) and surface \( S \). Suppose that the eigenvalues (1.1) are given for the eigenvalue equation \( (\Delta^2 + \lambda) u = 0 \) in \( \Omega \) together with the impedance boundary conditions:

\[
\left(\frac{\partial}{\partial r} + \gamma_j\right) u = 0 \quad \text{on} \quad S_j, \quad j = 1, \ldots, J
\]

where the surface \( S \) consists of parts \( S_j, \quad j = 1, \ldots, J \) such that
The object of this paper is to determine the geometry of the domains in (P1) and (P2) as well as the impedances $\gamma_j$, $j = 1, \ldots, J$ from the asymptotic expansion of the spectral function

$$\theta(t) = \sum_{m=1}^{\infty} \exp(-t \lambda_m), \tag{1.4}$$

for small positive $t$.

Zayed [1] has recently investigated problems (P1) and (P2) in the special case when $J = 2$, that is, when the boundary $\Gamma$ consists of two parts $\Gamma_1$, $\Gamma_2$ and when the surface $S$ consists of two parts $S_1$, $S_2$. Finally, we close this introduction with the remark that the author [2,3] has recently generalized the results of [1] to the case when $\Omega \subset \mathbb{R}^n$, $n = 2$ or $3$ is a simply connected bounded domain with a smooth boundary.

2. CONSTRUCTION OF $\theta(t)$ FOR PROBLEM (P1).

Following the method of Kac [4] and following closely the procedure of section 2 in Zayed [1], it is easy to show that the spectral function (1.4) associated with problem (P1) is given by:

$$\theta(t) = \iint_{\Omega} G(x, x'; t) \, dx, \tag{2.1}$$

where $G(x, x'; t)$ is the Green's function for the heat equation

$$\left( \Delta - \frac{3}{\rho^2} \right) u = 0, \tag{2.2}$$

subject to the impedance boundary conditions (1.2) and the initial condition

$$\lim_{t \to 0} G(x, x'; t) = \delta(x - x'), \tag{2.3}$$

where $\delta(x - x')$ is the Dirac delta function located at the source point $x = x'$. Let us write

$$G(x, x'; t) = G_0(x, x'; t) + x(x, x'; t), \tag{2.4}$$

where

$$G_0(x, x'; t) = (4\pi t)^{-1} \exp\left(-\frac{|x - x'|^2}{4t}\right), \tag{2.5}$$

is the "fundamental solution" of the heat equation (2.2), while $x(x, x'; t)$ is the "regular solution" chosen so that $G(x, x'; t)$ satisfies the impedance boundary conditions (1.2).

On setting $x = x'$ we find that

$$\theta(t) = \text{area} \frac{\Omega}{4\pi} + K(t), \tag{2.6}$$

where

$$K(t) = \iint_{\Omega} x(x, x'; t) \, dx. \tag{2.7}$$
The problem now is to determine the asymptotic expansion of $K(t)$ for small positive $t$. In what follows we shall use Laplace transform with respect to $t$ and use $s^2$ as the Laplace transform parameter; thus

$$G(x',x';t) = \int_{0}^{+\infty} e^{-s^2t} G(x',x';t) dt. \quad (2.8)$$

An application of the Laplace transform to the heat equation (2.2) shows that $\mathcal{G}(x',x';s^2)$ satisfies the two-dimensional membrane equation

$$\left(\Delta^2 - s^2\right) \mathcal{G}(x',x';s^2) = -\delta(x-x') \quad \text{in } \Omega, \quad (2.9)$$

together with the impedance boundary conditions (1.2). The asymptotic expansion of $K(t)$ as $t \to 0$, may then be deduced directly from the asymptotic expansion of $\mathcal{K}(s^2)$ for $s \to \infty$, where

$$\mathcal{K}(s^2) = \iint_{\Omega} \mathcal{G}(x,x';s^2) dx. \quad (2.10)$$

With reference to section 3 in Stewartson and Waechter [5], it can readily be shown after some reduction that the impedance boundary conditions (1.2) give

$$\mathcal{K}(s^2) = \frac{2}{4\pi} \sum_{m=-\infty}^{\infty} \left\{ \sum_{j=1}^{J} (\alpha_{j+1} - \alpha_j) f_j(m;s) \right\}, \quad (2.11)$$

where

$$f_j(m;s) = \left(1 + \frac{m}{2a^2} \right) \frac{I_m(sa)K_m(sa)}{a[I_m'(sa) + \gamma_j I_m(sa)]} - \frac{I'(sa)K'(sa)}{sa[I_m'(sa) + \gamma_j I_m(sa)]}, \quad (2.12)$$

in which $I_m$ and $K_m$ are modified Bessel functions. The series (2.11) is slowly convergent for large positive $s$ and it is therefore, expedient to apply a Watson transformation [5] to obtain

$$\mathcal{K}(s^2) = -\frac{2}{2\pi} \sum_{j=1}^{J} (\alpha_{j+1} - \alpha_j) \int_{0}^{+\infty} f_j(v;s) dv = as s \to \infty. \quad (2.13)$$

It now follows that the functions $f_j(v;s)$, $j = 1, \ldots, J$ may be expressed in terms of the asymptotic expansions of the modified Bessel functions and their derivatives due to Olver [6]. These expansions for $s \to \infty$ are uniformly valid in $v$ for $|\arg v| < \frac{\pi}{2}$.

Now, the following cases can be considered:

CASE 1. ($0 < \alpha_j \ll 1, j = 1, \ldots, J$)

In this case, it can be shown for $s \to \infty$ that

$$f_j(v;s) \sim \frac{(v+sa^2)^{1/2}}{s^{1/2}a^2} \sum_{n=0}^{\infty} \frac{A_{jn}(\tau)}{v^n}, \quad (2.14)$$

where $\tau = \frac{v}{(v+sa^2)^{1/2}}$. For $n = 0, 1, 2, 3$ we deduce that
\[ A_{j,0} = 0, \quad A_{j,1} = -\frac{1}{2}(r^2 - 3), \quad A_{j,2} = \frac{1}{2}(a\gamma_j - \frac{1}{2}) - \frac{r}{4}(a\gamma_j - \frac{3}{2}) - \frac{3}{2}, \]
and
\[ A_{j,3} = -\frac{3}{8}(3 - a\gamma_j + a\gamma_j^2) - \frac{r}{4}(\frac{23}{8} - 3a\gamma_j - a\gamma_j^2) - \frac{r}{4}(\frac{41}{8} - 2a\gamma_j) + \frac{21}{8} r. \quad (2.15) \]

On inserting (2.14) into (2.13) we deduce after some simplification that
\[ K(s^2) = \frac{\text{length} \sum_{j=1}^{J} \left( a_{j+1} - a_j \gamma_j \right) \frac{1}{6s} + O(1)}{s^2} \quad \text{as } s \to \infty. \quad (2.16) \]

On inverting Laplace transforms and using (2.6) we have the formula:
\[ \theta(t) = \frac{\text{area} \Omega}{4\pi t} + \frac{1}{8(\pi t)^{1/2}} \left[ \sum_{j=1}^{k} (a_{j+1} - a_j \gamma_j) \right] \frac{1}{6} + O(t^{1/2}) \quad \text{as } t \to 0. \quad (2.17) \]

CASE 2. \((0 < \gamma_j << 1, \ j = 1, ..., k \ \text{and} \ \gamma_j >> 1, \ j = k+1, ..., J)\)

In this case \(f_j(v;s), j = 1, ..., k\) have the same forms (2.14) and (2.15) while \(f_j(v;s), j = k+1, ..., J\) have the form (2.14) where
\[ A_{j,0} = 0, \quad A_{j,1} = \frac{r}{2} + \frac{3}{4} \left( \frac{1}{a\gamma_j} \right) - \frac{1}{2} \frac{r}{2} \frac{5}{a\gamma_j}, \]
\[ A_{j,2} = -\frac{r}{8a\gamma_j} + \frac{4}{8a\gamma_j} \left( \frac{19}{8a\gamma_j} \frac{1}{2} \right) - \frac{r}{6} \left( \frac{43}{8a\gamma_j} \frac{1}{2} \right) + \frac{25}{8a\gamma_j} r, \]
and
\[ A_{j,3} = -\frac{3}{8a\gamma_j} - \frac{1}{8} \frac{r}{2} \frac{27}{4a\gamma_j} - \frac{13}{8} - \frac{r}{7} \frac{107}{4a\gamma_j} - \frac{27}{8} \]
\[ + \frac{r}{9} \left( \frac{141}{4a\gamma_j} - \frac{15}{8} \right) \frac{11}{a\gamma_j}. \quad (2.18) \]

Consequently, we deduce after some reduction that
\[ \theta(t) = \frac{\text{area} \Omega}{4\pi t} + \frac{1}{8(\pi t)^{1/2}} \left[ a \sum_{j=1}^{k} (a_{j+1} - a_j \gamma_j) \frac{1}{6} \right] \frac{1}{6} + O(t^{1/2}) \quad \text{as } t \to 0. \quad (2.19) \]

CASE 3. \((\gamma_j >> 1, \ j = 1, ..., k \ \text{and} \ 0 < \gamma_j << 1, \ j = k+1, ..., J)\)

This case can be deduced from the previous one and yields:
\[ \theta(t) = \frac{\text{area} \Omega}{4\pi t} + \frac{1}{8(\pi t)^{1/2}} \left[ a \sum_{j=1}^{J} (a_{j+1} - a_j \gamma_j) \frac{1}{6} \right] \frac{1}{6} + O(t^{1/2}) \quad \text{as } t \to 0. \quad (2.20) \]

CASE 4. \((\gamma_j >> 1, \ j = 1, ..., J)\)

In this case \(f_j(v;s), j = 1, ..., J\) have the same forms (2.14) and (2.18). Consequently we have the formula:
\[ \theta(t) = \frac{\text{area} \Omega}{4\pi t} + \frac{1}{8(\pi t)^{1/2}} \left[ a \sum_{j=1}^{J} (a_{j+1} - a_j \gamma_j) (a+\gamma_j) \frac{1}{6} \right] \frac{1}{6} + O(t^{1/2}) \quad \text{as } t \to 0. \quad (2.21) \]

With reference to section 1 in Zayed [1] and the articles by Kac [4], Gottlieb
[7], Pleijel [8], and Sleeman and Zayed [9], the asymptotic expansions (2.17), (2.19), (2.20) and (2.21) may be interpreted as:

1. If \( \Omega \) is a circular domain of radius \( a \) and we have the impedance boundary conditions (1.2) with small/large impedances \( \gamma_j \), \( j = 1, \ldots, J \) as indicated in the specifications of the four respective cases, or (ii) for the first three terms, \( \Omega \) is a bounded domain in \( \mathbb{R}^2 \) of area \( \pi a^2 \). Let \( h < \infty \) be the number of smooth convex holes in \( \Omega \).

In case 1, it has \( n = \frac{1}{\pi} \sum_{j=1}^{J} (\alpha_{j} + 1 - \alpha_j) \gamma_j \) holes and a boundary length of

\[ 2\pi a \]

2 together with Neumann boundary conditions, provided \( h \) is an integer.

In case 2, it has \( h = \frac{2a}{\pi} \sum_{j=1}^{J} (\alpha_{j+1} - \alpha_j) \gamma_j \) holes, the parts \( \Gamma_j, j = 1, \ldots, k \) of the boundary \( \Gamma \) have lengths \( \frac{1}{\pi} \sum_{j=1}^{k} (\alpha_{j+1} - \alpha_j) \) together with Neumann boundary conditions while the other parts \( \Gamma_j, j = k+1, \ldots, J \) have lengths \( \frac{1}{\pi} \sum_{j=k+1}^{J} (\alpha_{j+1} - \alpha_j) (a + \gamma_j^{-1}) \) together with Dirichlet boundary conditions.

In case 4, it has no holes (\( h = 0 \)) and a boundary length of \( \frac{1}{\pi} \sum_{j=1}^{J} (\alpha_{j+1} - \alpha_j) \gamma_j \) together with Dirichlet boundary conditions.

We close this section with the remark that when \( J = 2 \) the results (2.17), (2.19), (2.20) and (2.21) are in agreement with the results of [1].

3. CONSTRUCTION OF \( \Theta(t) \) FOR PROBLEM (P2).

In analogy with the two dimensional membrane problem, it is clear that \( \Theta(t) \) associated with problem (P2) is given by:

\[ \Theta(t) = \iint_{\Omega} G(x,x';t)dx, \]

(3.1)

where \( G(x,x';t) \) is the Green's function for the heat equation

\[ (\Delta_3 - \frac{\partial}{\partial t}) u = 0, \]

(3.2)

subject to the impedance boundary conditions (1.3) and the initial condition of the form (2.3). As we have done in section 2, we can write \( G(x,x';t) \) for problem (P2) in a form similar to (2.4), where

\[ G(x,x';t) = (4\pi t)^{-3/2} \exp\left(-\frac{|x - x'|^2}{4t}\right). \]

(3.3)

From (2.4), (3.1) and (3.3) we find that

\[ \Theta(t) = \text{volume } \Omega + K(t) \]

(3.4)

where

\[ K(t) = \iint_{\Omega} x(x,x';t)dx. \]

(3.5)

An application of the Laplace transform to the heat equation (3.2) shows that

\[ \tilde{G}(x,x';s^2) = \tilde{\Theta}(x - x') \]

in \( \Omega \),

(3.6)
together with the impedance boundary conditions (1.3), where

$$
\overline{K}(s^2) = \iint \frac{\overline{x}(x, x; s^2)}{s^2} dx.
$$

With reference to section 2 in Waechter [10], it can readily be shown after some
reduction that the impedance boundary conditions (1.3) give

$$
\overline{K}(s^2) = -\frac{a^2}{2\pi} \sum_{m=0}^{\infty} (m + \frac{1}{2})^2 \left\{ \sum_{j=1}^{J} (\alpha_{j+1} - \alpha_j) f_j(m; s) \right\},
$$

where $f_j(m; s)$ have the same form (2.12) with $m$ replaced by $m + \frac{1}{2}$.

The series (3.8) if fact diverges since $K(t) \sim \frac{1}{t}$ for small positive $t$; however,
this difficulty may be easily removed by considering the asymptotic expansion for
large positive $s$ of

$$
\overline{K}_N(s^2) = -\frac{a^2}{2\pi} \sum_{m=0}^{N} (m + \frac{1}{2})^2 \left\{ \sum_{j=1}^{J} (\alpha_{j+1} - \alpha_j) f_j(m; s) \right\}.
$$

Inversion of the Laplace transform gives $K_N(t)$ and we may then write

$$
K(t) = \lim_{N \to \infty} K_N(t).
$$

On applying a Watson transformation [10] to (3.9), we find that

$$
\overline{K}_N(s^2) \sim -\frac{a^2}{2\pi} \sum_{j=1}^{J} (\alpha_{j+1} - \alpha_j) \int_{0}^{\infty} f_j(v; s) dv \quad \text{as} \quad s \to \infty.
$$

Now, the four respective cases considered in section 2, can be applied as
follows:

**CASE 1.** ($0 < \gamma_j << 1$, $j = 1, \ldots, J$)

On inserting (2.14) and (2.15) into (3.11) and integrating and letting $N \to \infty$, we
deduce after some simplification that

$$
K(t) = \frac{\text{surface area} S}{16\pi t} + \frac{1}{12\pi^\frac{3}{2} t^{\frac{1}{2}}} \left\{ 2a^2 \sum_{j=1}^{J} (\alpha_{j+1} - \alpha_j) \left( \frac{1}{a} - 3\gamma_j \right) \right\}
+ O(t^{\frac{1}{2}}) \quad \text{as} \quad t \to 0.
$$

**CASE 2.** ($0 < \gamma_j << 1$, $j = 1, \ldots, k$ and $\gamma_j >> 1$, $j = k+1, \ldots, J$)

On inserting (2.14), (2.15) and (2.18) into (3.11) and integrating and letting
$N \to \infty$ we have the formula

$$
\theta(t) = \frac{\text{volume} \Omega}{(4\pi t)^{\frac{3}{2}}} + \frac{1}{16\pi} \frac{1}{12\pi^\frac{3}{2} t^{\frac{1}{2}}} \left\{ 2a^2 \sum_{j=1}^{J} (\alpha_{j+1} - \alpha_j) \left( \frac{1}{a} - 3\gamma_j \right) \right\}
+ O(t^{\frac{1}{2}}) \quad \text{as} \quad t \to 0.
$$
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\[ + \frac{1}{12\pi^{3/2}t^{1/2}} \left(2a^{2} \sum_{j=1}^{k} (a_{j+1} - a_{j})(\frac{1}{a} - 3\gamma_{j}) + 2a \sum_{j=k+1}^{J} (a_{j+1} - a_{j}) \right) \]

\[ + 0(t^{1/2}) \quad \text{as } t \to 0. \]  

(CASE 3. \( \gamma_{j} \gg 1, j = 1, \ldots, k \) and \( 0 < \gamma_{j} \ll 1, j = k+1, \ldots, J \))

This case can be deduced from the previous one and yields

\[ \theta(t) = \frac{\text{volume } \Omega}{(4\pi t)^{3/2}} + \frac{1}{16\pi} \left(2a^{2} \sum_{j=1}^{k} (a_{j+1} - a_{j}) - 2a \sum_{j=1}^{k} (a_{j+1} - a_{j})(a - 2\gamma_{j}^{-1}) \right) \]

\[ + \frac{1}{12\pi^{3/2}t^{1/2}} \left(2a^{2} \sum_{j=k+1}^{J} (a_{j+1} - a_{j})(\frac{1}{a} - 3\gamma_{j}) + 2a \sum_{j=k+1}^{J} (a_{j+1} - a_{j}) \right) \]

\[ + 0(t^{1/2}) \quad \text{as } t \to 0. \]  

(CASE 4. \( \gamma_{j} \gg 1, j = 1, \ldots, J \))

On inserting (2.14) and (2.18) into (3.11) and integrating and letting \( N \to \infty \) we have the formula

\[ \theta(t) = \frac{\text{volume } \Omega}{(4\pi t)^{3/2}} - \frac{1}{16\pi} \left(2a \sum_{j=1}^{k} (a_{j+1} - a_{j})(a - 2\gamma_{j}^{-1}) \right) + \frac{a}{3(\pi t)^{1/2}} + 0(t^{1/2}) \]

as \( t \to 0. \)  

With reference to section 1 in [1] and the articles by Gottlieb [7], Waechter [10], Pleijel [11], and Zayed [12] the asymptotic expansions (3.13) - (3.16) may be interpreted as (i) \( \Omega \) is a spherical domain of radius \( a \) and we have the impedance boundary conditions (1.3) with small/large impedances \( \gamma_{j}, j = 1, \ldots, J \) as indicated in the specifications of the four respective cases, or (ii) for the first three terms, \( \Omega \) is a bounded domain in \( \mathbb{R}^{3} \) of volume \( \frac{4}{3}\pi a^{3} \).

In case 1, it has a surface \( S \) of area \( 4\pi a^{2} \), the parts \( S_{j}, j = 1, \ldots, J \) of the surface \( S \) have areas \( 2a^{2} \sum_{j=1}^{J} (a_{j+1} - a_{j}) \) and mean curvatures \( (\frac{1}{a} - 3\gamma_{j}) \), \( j = 1, \ldots, J \) together with Neumann boundary conditions.

In case 2, the parts \( S_{j}, j = 1, \ldots, k \) of the surface \( S \) have areas

\[ 2a^{2} \sum_{j=1}^{k} (a_{j+1} - a_{j}) \] 

and mean curvatures \( (\frac{1}{a} - 3\gamma_{j}) \), \( j = 1, \ldots, k \) together with Neumann boundary conditions, while the other parts \( S_{j}, j = k+1, \ldots, J \) have areas

\[ 2a \sum_{j=k+1}^{J} (a_{j+1} - a_{j})(a - 2\gamma_{j}^{-1}) \] 

and mean curvature \( \frac{1}{a} \) together with Dirichlet boundary conditions.

In case 4, it has a surface of area \( 2a \sum_{j=1}^{J} (a_{j+1} - a_{j})(a - 2\gamma_{j}^{-1}) \) and mean curvature \( \frac{1}{a} \) together with Dirichlet boundary conditions.
Finally, we note that when $J = 2$ the results (3.13) - (3.16) are in agreement with the results of [1].

4. DISCUSSIONS.

This paper represents a sensible extension of the author's previous publication [1] when the boundary $\Gamma$ in $\mathbb{R}^2$ or the surface $S$ in $\mathbb{R}^3$ consists of two parts ($J = 2$) to the case when $\Gamma$ or $S$ consists of $J$ parts, where $J$ is a finite positive integer, in which a great deal of technical analysis has gone into obtaining the results. Zayed [2,3] has recently generalized the results of [1] to the case when $\Omega \subseteq \mathbb{R}^n$, $n = 2$ or $3$ is a simply connected bounded domain, where a considerable amount of mathematical work has gone into obtaining the results. With reference to the previous work (See [2], [3], [11], [12]), we conclude that, there are technical difficulties and a considerable amount of mathematical work in extending the results of the present paper to the type of domains considered in [2] and [3]. This extension is still an open problem which will be discussed in a forthcoming paper.

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