ABSTRACT. In this paper we define and develop a theory of differentiation in Wiener space C[0,T]. We then proceed to establish a fundamental theorem of the integral calculus for C[0,T]. First of all, we show that the derivative of the indefinite Wiener integral exists and equals the integrand functional. Secondly, we show that certain functionals defined on C[0,T] are equal to the indefinite integral of their Wiener derivative.

KEY WORDS AND PHRASES. Wiener (measure, integral, derivative, absolute continuity), Lebesgue absolute continuity, fundamental theorem.

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1. INTRODUCTION.

Consider the Wiener measure space \((C[0,T], \mathcal{F}^*, m_w)\) where C[0,T] is the space of all continuous functions x on \([0,T]\) vanishing at the origin. For each partition \(\tau = \tau_n = \{t_0, \ldots, t_n\}\) of \([0,T]\) with \(0 = t_0 < t_1 < \cdots < t_n = T\), let \(X_\tau: C[0,T] \to \mathbb{R}^n\) be defined by \(X_\tau(x) = x(\tau) = (x(t_1), \ldots, x(t_n))\). Let \(\mathcal{B}^n\) be the \(\sigma\)-algebra of Borel sets in \(\mathbb{R}^n\). Then a set of the type
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\[ I = \{ x \in C[0,T] : X_\tau(x) \in B \} \equiv X^{-1}_\tau(B), \ B \in \mathcal{B}^n \]

is called a Wiener interval (or a Borel cylinder). It is well known that

\[ m_w(I) = \int_B K(\tau, \eta) d\eta, \quad (1.1) \]

where

\[ K(\tau, \eta) = \left\{ \prod_{j=1}^{n} \frac{2\pi(t_j-t_{j-1})^2}{\tau_j} \right\}^{-1/2} \exp \left\{ -\frac{1}{2} \sum_{j=1}^{n} \frac{(\eta_j - \eta_{j-1})^2}{\tau_j} \right\} \quad (1.2) \]

with \( \eta = (\eta_1, \ldots, \eta_n) \), and \( \eta_0 = 0 \). The measure \( m_w \) is a probability measure defined on the algebra \( \mathcal{F} \) of all Wiener intervals and \( m_w \) is extended to the Caratheodory extension \( \mathcal{F}^* \) of \( \mathcal{F} \). Let \( \mathcal{F}_\tau \) be the \( \sigma \)-algebra generated by the set \( \{ X^{-1}_\tau(B) : B \in \mathcal{B}^n \} \) with \( \tau \) fixed. Then, by the definition of conditional expectation, see Doob [1], Tucker [2] and Yeh [3], for each Wiener integrable function \( F(x) \),

\[ \int_B E(F|\mathcal{F}_\tau) m_w(dx) = E(F) m_w(dx), \quad (1.3) \]

where \( P_{X_\tau}(d\eta), \ B \in \mathcal{B}^n \),

\[ \frac{dP_{X_\tau}}{d\eta} = K(\tau, \eta), \ \eta \in \mathbb{R}^n. \quad (1.5) \]

Next, for each \( F \in L_1(C[0,T],m_w) \) and each partition \( \tau \) of \( C[0,T] \), let

\[ F_\tau = E(F|\mathcal{F}_\tau) \quad (1.6) \]

and

\[ \tilde{F}(\eta) = E(F(x)|X_\tau(x) = \eta) = E(F|X_\tau)(\eta) \quad (1.7) \]

Then, \( \{ F_\tau \} \) is a martingale, and by the martingale convergence theorem,

\[ \lim_{||\tau|| \to 0} F_\tau(x) = F(x) \quad (1.8) \]

for almost all \( x \in C[0,T] \). Furthermore,

\[ F(x) = \lim_{||\tau|| \to 0} E(F(y)|X_\tau(y) = x(\tau)) = \lim_{||\tau|| \to 0} \tilde{F}(x(\tau)) \quad (1.9) \]

for almost all \( x \in C[0,T] \).

For a given partition \( \tau = \tau_n \) of \([0,T]\) and \( x \in C[0,T] \), define the polygonal function \([x] \equiv [x(\tau)]\) on \([0,T] \) by
Similarly, for each \( \eta_j = (\eta_1, \cdots, \eta_n) \in \mathbb{R}^n \), define the polygonal function \( \tilde{\eta} \) of \( \eta \) on \([0,T]\) by
\[
\tilde{\eta}(t) = \eta_{j-1} + \frac{t-t_{j-1}}{t_{j}-t_{j-1}} (\eta_j - \eta_{j-1}), \quad t_{j-1} \leq t \leq t_j, \quad j = 1, \cdots, n
\]
with \( \eta_0 = 0 \).
Then both functions \( \eta \) and \( \tilde{\eta} \) are continuous on \([0,T]\), their graphs are line segments on each subinterval \([t_{j-1}, t_j]\), and \( \eta(t_j) = \eta_j \) and \( \tilde{\eta}(t_j) = \eta_j \) at each \( t_j \in \tau \).

For \( x, y \in C[0,T] \), we use the convention:
\[
x \leq y \quad \text{if and only if} \quad x(t) \leq y(t) \quad \text{for every} \quad t \in [0,T],
\]
and
\[
x < y \quad \text{if and only if} \quad x(t) < y(t) \quad \text{for every} \quad t \in (0,T].
\]

The main purpose of this paper is to define and develop a theory of differentiation in Wiener space \( C[0,T] \), and then to establish a fundamental theorem of the integral calculus on \( C[0,T] \); namely, that the Wiener derivative of the indefinite integral \( \int F(y)m_w(dy) \) is \( F(x) \), and that a Wiener absolutely continuous function can be expressed as the indefinite integral of its Wiener derivative. This study was initiated by Smolowitz [4]. In this paper we incorporate some recent results of Park and Skoug [5] to improve and substantially simplify the concepts and results of Smolowitz [4].

2. THE WIENER DERIVATIVE.

Our first objective is to define the Wiener derivative \( \partial_x(\cdot) \) so that
\[
\partial_x \int_{y \leq x} F(y)m_w(dy) = F(x)
\]
for \( F \in L_1(C[0,T],m_w) \). We start by quoting the following theorem from Park and Skoug [5] which plays an important role in this paper.

THEOREM A. Let \( F \in L_1(C[0,T],m_w) \). Then for any Borel set \( B \in \mathcal{B}^n \),
\[
\mu_{\tau}(B) = \int_{B} F(x)m_w(dx) = \int_{C[0,T]} E_{R}(F(x) - [x] + [\tilde{\eta}]) P_{X_{\tau}}(d\tilde{\eta}) \tag{2.1}
\]
where
\[
E_{R}[F(x - [x] + [\tilde{\eta}])] = \int_{C[0,T]} F(x - [x] + [\tilde{\eta}])m_w(dx).
\]

In view of (1.3) and (2.1), we may conclude that
\[
E(F(x)|X_{\tau}(x)) = \tilde{\eta} \quad \text{for almost all} \quad \tilde{\eta} \quad \text{in} \quad \mathbb{R}^n; \quad \text{i.e., we may express the conditional expectation} \quad E(F|X_{\tau})(\tilde{\eta}) \quad \text{in terms of an ordinary Wiener integral}. 
\]
Note that for \( F \in L_1(C[0,T],m_w) \),
\[
\tilde{F}(\tilde{\eta}) = E(F|X_{\tau})(\tilde{\eta}) \quad \text{is in} \quad L_1(\mathbb{R}^n, P_{X_{\tau}}(d\tilde{\eta})).
\]
Also note that for each \( x \in C[0,T] \) and each
partition \( \tau = \{t_1, \ldots, t_n\} \) of \([0,T]\), \( F(x(\tau)) = E(F(y)|X_\tau(y) = x(\tau)) \) is a function of \( x(t_1), \ldots, x(t_n) \).

**DEFINITION 1.** Let \( F \in L_1(C[0,T],m_w) \). For each partition \( \tau = \{t_1, \ldots, t_n\} \) of \([0,T]\) define the operator \( \mathcal{D}_x(\tau) \) by

\[
\mathcal{D}_x(\tau)F(x) = \frac{\partial^n F(x(\tau))}{\partial x(t_n) \cdots \partial x(t_1)} / K(\tau,x(\tau))
\]

if it exists. Furthermore, if \( \mathcal{D}_x(\tau)F(x) \) exists for each partition \( \tau \), then the Wiener derivative of \( F(x) \) is defined by

\[
\mathcal{D}_xF(x) = \lim_{||\tau|| \to 0} \mathcal{D}_x(\tau)F(x)
\]

if the limit exists.

Our first theorem is the first half of the fundamental theorem of Wiener calculus.

**THEOREM 1.** Let \( F \in L_1(C[0,T],m_w) \). Then

\[
\int_{y \leq x} F(y)m_w(dy) = F(x)
\]

for almost all \( x \in C[0,T] \).

**PROOF.** For \( x \in C[0,T] \) let \( G(x) \) denote the indefinite Wiener integral

\[
G(x) = \int_{y \leq x} F(y)m_w(dy) = E_y[I_x(y)F(y)]
\]

(2.4)

where \( I_x(y) \) is the indicator function

\[
I_x(y) = \begin{cases} 1, & y(t) \leq x(t) \text{ for all } t \in [0,T] \\ 0, & \text{otherwise.} \end{cases}
\]

Then using (1.7), (2.4), (2.2), (1.3), (2.2) and the Fubini theorem, we obtain

\[
\tilde{G}(\eta) = E(G(u)|X_\tau(u) = \eta)
\]

\[
= E_u(E_y[I_u(y)F(y)|X_\tau(u) = \eta])
\]

\[
= E_u[E_y[I_{u-}[y]+[\eta](y)F(y)]]
\]

\[
= E_u\left[ \int_{\mathbb{R}^n} E_y[I_{u-}[y]+[\eta](y)F(y)|X_\tau(y) = \xi]P_{X_\tau}(d\xi) \right]
\]

\[
= \int_{\mathbb{R}^n} E_u[E_y[I_{u-}[y]+[\eta](y-\xi)+[\xi]]F(y-\xi)|X_\tau(u)]P_{X_\tau}(d\xi)
\]

(2.5)

But \( I_{u-}[y]+[\eta](y-\xi)+[\xi] \) is zero unless

\[
y(t) - [y](t) + [\xi](t) \leq u(t) - [u](t) + [\eta](t)
\]

for all \( t \in [0,T] \). But (2.6) implies that

\[
\xi_j = y(t_j) - [y](t_j) + [\xi](t_j) \leq u(t_j) - [u](t_j) + [\eta](t_j) = \eta_j
\]

for \( j = 1, \ldots, n \). Hence we can write
\[ G(y) = \int_{-\infty}^{\eta_n} \cdots \int_{-\infty}^{\eta_1} \mathbb{E}_u [E_y [I_{u-[u]} \mathbb{E}_y (y-[y]+[\xi])] \cdot F(y-[y]+[\xi]) ] K(\tau, \xi) d\xi_1 \cdots d\xi_n, \]

and so for each \( x \in C[0,T], \)

\[ x(t_n) \]

Hence

\[ x(t_1) \]

\[ x(t_i) F(y) \mathbb{E}_y (y-[y]+[x(t)]) \]

Applying (1.9) to (2.7) yields

\[ \mathbb{D}_x (G(x)) = 1_{\lim_{\|\tau\| \to 0}} \mathbb{E}_{u,y} (I_u(y) F(y) | X(\tau) = x(\tau), X(\tau(u) = x(\tau)) ) K(\tau, x(\tau)). \]

for almost all \( x \) in \( C[0,T] \) which concludes the proof of Theorem 1.

COROLLARY 1. If \( \{t_1, \ldots, t_n\} \subseteq \tau = \{t_1, \ldots, t_n\} \) and if \( F(y) = f(y(t_1), \ldots, y(t_n)) \)

is in \( L_1(C[0,T], m_w) \), then

\[ \mathbb{D}_x (G(x)) \]

\[ \mathbb{D}_x (G(x)) \]

and

\[ \mathbb{D}_x (G(x)) \]

for almost all \( x \) in \( C[0,T] \).

PROOF. Using (2.7) and (2.4) we see that

\[ \mathbb{D}_x (G(x)) \]

Under the conditioning \( X(\tau) = x(\tau), F(y) \) becomes \( f(x(t_1), \ldots, x(t_n)) \) which equals \( F(x) \).

Therefore,

\[ \mathbb{D}_x (G(x)) \]

As \( \|\tau\| \to 0 \), \( E_{u,y} (I_u(y) | X(\tau) = x(\tau)) \) \( = I_x(x) = 1 \) by (1.9) for almost all \( x \) in \( C[0,T] \). Thus Corollary 1 is established.

COROLLARY 2. Let \( \tau' = \{t_1, \ldots, t_n\} \) be any partition of \([0,T]\), and let

\[ F(x) = f(x(t_1), \ldots, x(t_n)) \] be in \( L_1(C[0,T], m_w) \). Then \( \mathbb{D}_x F(x) = 0. \)
PROOF. Let \( \tau \) be a partition of \([0,T]\) properly containing \( \tau' \). Then
\[
\tilde{F}(x(\tau)) = E(F|y)|X_{\tau}(y) = x(\tau)) = f(x(t_1'), \ldots, x(t_n')).
\]
Thus \( \mathcal{A}_\tau F(x) = 0 \), and so \( \mathcal{A}_\tau F(x) = 0 \).

3. LEBESGUE AND WIENER ABSOLUTE CONTINUITY.

In this section we show that certain functions defined on \( C[0,T] \) are equal to the indefinite integral of their Wiener derivative.

For \( \vec{a} = (a_1, \ldots, a_n) \) and \( \vec{b} = (b_1, \ldots, b_n) \) in \( \mathbb{R}^n \) with \( a_i < b_i \), \( i = 1, \ldots, n \), let \( \mathcal{V}(\vec{a}, \vec{b}, k) \) be the collection of all points of the form \( \vec{v} = (v_1, \ldots, v_n) \) where each \( v_i \) is either \( a_i \) or \( b_i \) and exactly \( k \) of the \( v_i \) are \( a_i \)'s. For any function \( f \) defined on \( \mathcal{V}(\vec{a}, \vec{b}, k) \) for \( k = 0, 1, \ldots, n \), let
\[
\Delta_{\vec{a}, \vec{b}} f = f(\vec{b}) - f(\vec{a}) = \sum_{k=1}^{n} (-1)^k \sum_{\vec{v} \in \mathcal{V}(\vec{a}, \vec{b}, k)} f(\vec{v}) \quad (3.1)
\]

A function of \( n \) variables \( f(u_1, \ldots, u_n) \) is said to be Lebesgue absolutely continuous in the sense of Vitali (see Clarkson and Adams [6,7] and Hobson [8]) on the region \( \Omega \subset \mathbb{R}^n \) if, given \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that if \( I^*_k = \bigtimes_{i=1}^{n} (a_i^{(k)}, b_i^{(k)}) \), \( k = 1, 2, \ldots \), are disjoint \( n \)-dimensional rectangles contained in \( \Omega \) with \( \sum_{k=1}^{N} m_L(I^*_k) \leq \delta \) for any \( N \), then
\[
\sum_{k=1}^{N} |\Delta_{\vec{a}^{(k)}, \vec{b}^{(k)}} f| < \varepsilon, \quad m_L(\cdot) \text{ denotes } n \text{-dimensional Lebesgue measure, and}
\]

\( \vec{a}^{(k)} = (a_1^{(k)}, \ldots, a_n^{(k)}) \). A function \( f(u_1, \ldots, u_n) \) is said to be Lebesgue absolutely continuous (in the sense of Hardy-Krause; see Berkson and Gillespie [9], and Clarkson and Adams [6,7]) on a region \( \Omega \subset \mathbb{R}^n \) if for each \( k = 1, \ldots, n-1 \), whenever \( n-k \) variables are fixed then \( f \), as a function of its remaining \( k \) variables, is Lebesgue absolutely continuous in the sense of Vitali on \( \Omega \cap \mathbb{R}^k \). When we merely state "Lebesgue absolutely continuous", it is always meant in the sense of Hardy-Krause.

It is well known that if \( f(u_1, \ldots, u_n) \) is Lebesgue absolutely continuous in the sense of Vitali on \( \Omega \equiv \bigtimes_{i=1}^{n} [a_i, b_i] \), then \( \partial^R f(u_1, \ldots, u_n)/\partial u_1 \ldots \partial u_n \) exists a.e. on \( \Omega \) and is integrable on \( \Omega \). Furthermore
\[
\int_{\Omega} \partial^R f(u_1, \ldots, u_n)/\partial u_1 \ldots \partial u_n du_1 \ldots du_n = \Delta_{\vec{a}, \vec{b}} f, \quad (3.2)
\]
and
\[
\int_{\Omega} \partial^R f(u_1, \ldots, u_n)/\partial u_1 \ldots \partial u_n du_1 \ldots du_n = \text{Var}(f, R) \quad (3.3)
\]
where \( \text{Var}(f, R) \) denotes the total variation of \( f \) over \( R \).
Let $G(x)$ be any Wiener integrable function on $C[0,T]$. Then, by definition,

$$
\tilde{G}(\eta) = E(G|\mathcal{F}_\tau)(\eta) \quad \text{is a function of } \eta \text{ which is integrable with respect to}
$$

$$
P_X(d\eta) = K(\tau,\eta)d\eta.$$

**DEFINITION 2.** A Wiener integrable function $G(x)$ defined on $C[0,T]$ is said to be Wiener absolutely continuous provided that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if the sequence $\{ x \in C[0,T] : \alpha_i^{(k)} < x(s_i^{(k)}) \leq \beta_i^{(k)} \}$ with $-\infty \leq \alpha_i^{(k)} < \beta_i^{(k)} \leq \infty$ are disjoint Wiener intervals with $\sum_{k=1}^{N} m_w(I_k) < \delta$ for any $N$, then

$$
\sum_{k=1}^{N} \frac{\Delta_{\alpha_i^{(k)},\beta_i^{(k)}}G}{m_w(I_k)} \sim |\eta| < \varepsilon.
$$

The following propositions can be easily established.

**PROPOSITION A.** If $G(x)$ is Wiener absolutely continuous on $C[0,T]$, then for every partition $\tau$ of $(0,T]$, $\tilde{G}(\eta)$ is Lebesgue absolutely continuous on $\tau$, where $|\tau|$ denotes the number of points in $\tau$.

**PROPOSITION B.** Let $F \in L_1(C[0,T],m_w)$. Then the indefinite Wiener integral

$$
G(x) = \int_{y \leq x} F(y)m_w(dy)
$$

is Wiener absolutely continuous on $C[0,T]$.

Our next theorem is the second half of the fundamental theorem of Wiener Calculus.

**THEOREM 2.** Let $G \in L_1(C[0,T],m_w)$ satisfy the conditions:

(i) $\mathcal{D}_x G(x)$ exists for almost all $x \in C[0,T]$ and belongs to $L_1(C[0,T],m_w)$,

(ii) $G(x)$ is Wiener absolutely continuous on $C[0,T]$,

(iii) If $\{x_k\}_{k=1}^{\infty}$ is a sequence in $C[0,T]$ such that $x_k(s_0) \to -\infty$ as $k \to \infty$ for some fixed point $s_0 \in (0,T]$, then $G(x_k) \to 0$ as $k \to \infty$.

Then

$$
G(u) = \int_{x \leq u} \mathcal{D}_x G(x)m_w(dx)
$$

for almost all $u$ in $C[0,T]$.

**PROOF.** For given $\varepsilon > 0$ let $\delta = \delta(\varepsilon/3) > 0$ be the value for the Wiener absolute continuity of $G(x)$, and also assume that

$$
m_w(S) < \delta \Rightarrow \int_S |\mathcal{D}_x G(x)|m_w(dx) < \varepsilon/3.
$$

Let $\{\tau^{(k)}\}$ be a sequence of partitions of $[0,T]$ such that $\|\tau^{(k)}\| \to 0$ as $k \to \infty$. Then

$$
\lim_{k \to \infty} \mathcal{D}_{x(\tau^{(k)})} G(x) = \mathcal{D}_x G(x) \quad \text{for almost all } x \in C[0,T].
$$

By Egoroff's theorem, there exists a set $C \subset C[0,T]$ with $m_w(C) > 1 - \delta/2$ and a positive integer $k_0$ such that if $k \geq k_0$, then
Let
\[ C_k = \{ x \in C[0,T] : |D_x(r(k)) G(x) - D_x G(x)| < \varepsilon/3 \}, \]
for every \( x \in C_\varepsilon \).

Then \( C_\varepsilon \subseteq C_k \), and hence \( m_w(C_k) \geq m_w(C_\varepsilon) > 1 - \delta/2 \), and
\[
\int_{C_k} |D_x(r(k)) G(x) - D_x G(x)| m_w(dx) < \varepsilon/3 \quad \text{for } k \geq k_0.
\]
The complements satisfy \( m_w(C_k^c) \leq m_w(C_\varepsilon^c) < \delta/2 \) for \( k \geq k_0 \). Next consider fixed \( k \), \( k \geq k_0 \) and let \( q \) denote the number of points in the partition \( r(k) \). Let
\[ E_k = \{ \eta = (\eta_1, \ldots, \eta_q) \in \mathbb{R}^q : \eta = r(k) \text{ for some } x \in C_k^c \} . \]

Then,
\[
m_w(C_k^c) = \int_{E_k} K(r(k),\eta) d\eta \quad .
\]
Since \( K(r(k),\eta) \) is bounded in \( \eta \) on \( \mathbb{R}^q \) and
\[
\int_{E_k} K(r(k),\eta) d\eta = m_w(C_k^c) < \delta/2 \quad ,
\]
we can find a countable sequence of disjoint \( q \)-dimensional rectangles \( I_\ell^* = \bigcup_{i=1}^q (a_i^\ell, b_i^\ell) \), \( \ell = 1, 2, \ldots \)
such that \( E_k \subseteq \bigcup_{\ell=1}^\infty I_\ell^* \) and
\[
\int_{E_k} K(r(k),\eta) d\eta \leq \sum_{\ell=1}^\infty \int_{I_\ell^*} K(r(k),\eta) d\eta = \sum_{\ell=1}^\infty m_w(I_\ell) < \delta \quad ,
\]
where
\[
I_\ell = \{ x \in C[0,T] : a_i^\ell < x(s_i^\ell) \leq b_i^\ell \text{ for each } s_i^\ell \in r(k) \} \quad .
\]

Hence
\[
\int_{C_k^c} |D_x(r(k)) G(x)| m_w(dx) = \int_{E_k} |D^\ell \tilde{G}(\eta_1, \ldots, \eta_q) / \partial \eta_1 \cdots \partial \eta_q| d\eta_1 \cdots d\eta_q
\]
\[
\leq \sum_{\ell=1}^\infty \int_{a_i^\ell}^{b_i^\ell} \int_{a_i^\ell}^{b_i^\ell} |D^\ell \tilde{G}(\eta_1, \ldots, \eta_q) / \partial \eta_1 \cdots \partial \eta_q| d\eta_1 \cdots d\eta_q
\]
\[
= \sum_{\ell=1}^\infty \text{Var}(\tilde{G}, I_\ell^*) \quad ,
\]
where the last equality follows from (3.3). Now,
\[
\text{Var}(\tilde{G}, I_\ell^*) = \sup_i \Delta_{i, \sigma_1, \rho_1} \tilde{G}(\tilde{\eta})
\]
where the supremum is taken over all possible nets of \( I_\ell^* \), and each net has total Lebesgue
measure equal to that of $I^*_p$, and so the corresponding Wiener intervals have total measure equal to $m_w(I_p)$. Since $\sum_{\ell=1}^{\infty} m_w(I^*_\ell) < \delta(\varepsilon/3)$, by the Wiener absolute continuity of $G$, we have

$$\sum_{\ell=1}^{N} \sum_{i} |\Delta_{\hat{\eta}^\ell_i} G(\hat{\eta})| < \varepsilon/3$$

for every $N$ and every net.

Thus, by taking the supremum over all nets, we get

$$\sum_{\ell=1}^{N} \text{Var}(G, I^*_\ell) \leq \varepsilon/3$$

and hence

$$\int_{C_k^\sim} |D_{x(r(k))} G(x)| m_w(dx) \leq \varepsilon/3.$$

Thus, for every $k \geq k_0$,

$$\int_{C_k^\sim} |D_{x(r(k))} G(x) - D_{x} G(x)| m_w(dx) \leq \varepsilon/3$$

$$+ \int_{C_k^\sim} |D_{x(r(k))} G(x)| m_w(dx) \leq \varepsilon.$$ 

In particular

$$\int_{[x] \subseteq [u]} |D_{x(r(k))} G(x) - D_{x} G(x)| m_w(dx) < \varepsilon$$

for $k \geq k_0$, where $[\cdot]$ corresponds to $r(k)$. Hence

$$\lim_{k \to \infty} \left[ \int_{[x] \subseteq [u]} D_{x(r(k))} G(x)m_w(dx) - \int_{[x] \subseteq [u]} D_{x} G(x)m_w(dx) \right] = 0.$$ 

Since $\{x \in C[0,T] : [x] \subseteq [u]\} \rightarrow \{x \in C[0,T] : x \subseteq u\}$ as $k \to \infty$, an application of the dominated convergence theorem yields

$$\lim_{k \to \infty} \int_{[x] \subseteq [u]} D_{x} G(x)m_w(dx) = \int_{x \subseteq u} D_{x} G(x)m_w(dx).$$

Thus,

$$\lim_{k \to \infty} \int_{[x] \subseteq [u]} D_{x(r(k))} G(x)m_w(dx) = \int_{x \subseteq u} D_{x} G(x)m_w(dx).$$

On the other hand, using (2.3), (3.2) and (3.1), we see that for any $a \in C[0,T]$ with $a < u$,
\[
\int_{[a] \leq x \leq [u]} \mathcal{D}_x(\tau(k)) G(x)m_w(dx) = \int_{a(\tau(k)) \leq x(\tau(k)) \leq u(\tau(k))} \mathcal{D}_x(\tau(k)) G(x)m_w(dx)
\]

\[
\begin{align*}
&= \int_a^{s(k)} \cdots \int_{s_1(k)}^u \left[ \mathcal{D}^q G(\eta_1, \cdots, \eta_q)/\partial \eta_1 \cdots \partial \eta_q \right] d\eta_1 \cdots d\eta_q \\
&= \Delta_{a(\tau(k)), u(\tau(k))} \tilde{G} \\
&= \tilde{G}(u(\tau(k))) + \sum_{\ell=1}^{q} (-1)^{\ell} \sum_{\tilde{\nu} \in V(a(\tau(k)), u(\tau(k)), \ell)} \tilde{G}(\tilde{\nu}).
\end{align*}
\]

If we let \(a(s_i^{(k)}) \to -\infty\) as \(k \to \infty\) for \(i = 1, \cdots, q\) in (3.6), then by assumption (iii), \(\tilde{G}(\tilde{\nu}) \to 0\) as \(k \to \infty\) for every \(\tilde{\nu} \in V(a(\tau(k)), u(\tau(k)), \ell), \ell \geq 1\). Thus (3.6) reduces to

\[
\int_{[x] \leq u} \mathcal{D}_x(\tau(k)) G(x)m_w(dx) = \tilde{G}(u(\tau(k))).
\]

In view of (1.9) and (3.5), we conclude that

\[
\int_{x \leq u} \mathcal{D}_x G(x)m_w(dx) = G(u)
\]

for almost all \(u\) in \(C[0,T]\), and so (3.4) is established.

REFERENCES