SEPARATION PROPERTIES OF THE WALLMAN ORDERED COMPACTIFICATION

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(Received December 19, 1988 and in revised form February 8, 1990)

ABSTRACT. The Wallman ordered compactification \( w_0X \) of a topological ordered space \( X \) is \( T_2 \)-ordered (and hence equivalent to the Stone-Čech ordered compactification) iff \( X \) is a \( T_4 \)-ordered c-space. In particular, these two ordered compactifications are equivalent when \( X \) is \( n \) dimensional Euclidean space iff \( n \leq 2 \). When \( X \) is a c-space, \( w_0X \) is \( T_1 \)-ordered; we give conditions on \( X \) under which the converse statement is also true. We also find conditions on \( X \) which are necessary and sufficient for \( w_0X \) to be \( T_2 \). Several examples provide further insight into the separation properties of \( w_0X \).

KEY WORDS AND PHRASES. c-set, maximal c-filter, \( T_1 \)-ordered space, \( T_2 \)-ordered space, ordered compactification.

1980 AMS SUBJECT CLASSIFICATION CODES. 54F05, 54D35, 54D10

Introduction.

The Wallman ordered compactification \( w_0X \) of a \( T_1 \)-ordered space \( X \) was introduced in 1979 by Choe and Park [1]. In [3] one of the authors showed (in the terminology of this paper) that \( w_0X \) is \( T_2 \)-ordered iff \( X \) is a \( T_4 \)-ordered c-space, and that for such spaces, \( w_0X \) is equivalent to the Stone-Čech ordered (or Nachbin) compactification \( \beta_0X \) of \( X \).
This paper continues the study of the separation properties of $w_0 X$. If $X$ is a c-space (meaning that the increasing and decreasing hulls of every c-set are closed), then $w_0 X$ is $T_1$-ordered, and under certain further restrictions on $X$ the condition of being a c-space is shown to be necessary in order for $w_0 X$ to be $T_1$-ordered (see Theorems 2.7 and 2.8). Two conditions on $X$ are found which are necessary and sufficient for $w_0 X$ to be $T_2$; one is an ultrafilter condition, while the other is a version of normality for ordered spaces which we call "c-normally ordered." For $T_1$-ordered c-spaces, the notions "c-normally ordered" and "normally ordered" (as defined by Nachbin, [5]) are equivalent, but for $T_1$-ordered spaces in general it is shown by examples that neither property implies the other.

One motivation for studying the Wallman ordered compactification is that it gives a convenient filter characterization for $\beta_0 X$ when $X$ is a $T_1$-ordered c-space. For Euclidean n-space $R^n$, we show that $w_0 R^n$ and $\beta_0 R^n$ are equivalent iff $n \leq 2$, and we then give a description of $\beta_0 R^2$ based on the Wallman characterization of compactification points in $\beta_0 R^2$ as non-convergent maximal c-filters. Other examples are given to show how the separation properties of the Wallman ordered compactification can fail in various ways and combinations.

1. The Wallman Ordered Compactification.

If $(X, \leq)$ is a poset and $A$ a non-empty subset of $X$, we define $d(A) = \{y \in X : y \leq z$ for some $z \in A\}$ to be the decreasing hull of $A$; the increasing hull $i(A)$ is defined dually. We shall write $d(z)$ ($i(z)$) in place of $d(\{z\})$ ($i(\{z\})$). A subset $A$ is increasing (respectively, decreasing) if $A = i(A)$ (respectively, $A = d(A)$). A set which is either increasing or decreasing is said to be monotone. If $A = i(A) \cap d(A)$, then $A$ is called a conez set.

We shall use the term space throughout this paper to mean a triple $(X, \leq, r)$, where $(X, \leq)$ is a poset and $r$ a conez topology on $X$ (i.e., a topology for which the open monotone sets constitute an open subbase). When there is no danger of confusion, we shall designate the space $(X, \leq, r)$ simply by "$X$".

For any space $X$, we shall use the term fundamental open set to mean any set expressible as a finite intersection of finite unions of monotone open sets. The set $U_X$ of all fundamental open sets forms an open base for $X$. The complement of a fundamental open set will be called a fundamental closed set.

Let $A$ be a subset of a space $X$, and let $I(A)$ (respectively, $(D(A))$ be the smallest closed and increasing (respectively, closed and decreasing) set containing $A$, and let $A^\wedge = I(A) \cap D(A)$. If $A = A^\wedge$ then $A$ is called a c-set; let $C_X$ denote the collection of all c-sets on a space $X$. One can verify that $C_X$ is closed under
intersections and forms a subbase for the collection of closed sets in a space $X$. The relationship between fundamental open sets and c-sets can be described as follows.

**Proposition 1.1** Let $X$ be a space. Then $U \in \mathcal{U}_X$ if $X - U$ is a finite union of c-sets.

If $\mathcal{F}$ is a filter on a space $X$, let $I(\mathcal{F})$ be the filter generated by $\{I(F) : F \in \mathcal{F}\}$; the filters $D(\mathcal{F}), i(\mathcal{F})$, and $d(\mathcal{F})$ are defined similarly. The fixed ultrafilter generated by an element $x$ in $X$ is denoted by $\mathcal{F}_x$. If $\mathcal{F}$ and $\mathcal{G}$ are filters on $X$ which do not contain disjoint sets, let $\mathcal{F} \vee \mathcal{G}$ designate the filter generated by $\{F \cap G : F \in \mathcal{F}, G \in \mathcal{G}\}$; if $F \cap G = \emptyset$ for some $F \in \mathcal{F}$ and $G \in \mathcal{G}$, we say that $\mathcal{F} \vee \mathcal{G}$ "fails to exist" (as a proper filter).

For any filter $\mathcal{F}$, the filter $\mathcal{F}^\wedge = I(\mathcal{F}) \vee D(\mathcal{F})$ exists and is generated by $\{F^\wedge : F \in \mathcal{F}\}$. If $\mathcal{F} = \mathcal{F}^\wedge$, then $\mathcal{F}$ is called a c-filter. It is easy to show (using Zorn's Lemma) that every c-filter is coarser than a maximal c-filter. In our study of the Wallman ordered compactification, which is based on maximal c-filters, the following characterization will be useful.

**Proposition 1.2** A c-filter $\mathcal{F}$ on a space $X$ is maximal iff, for each c-set $A$, either $A \in \mathcal{F}$ or $X - A \in \mathcal{F}$.

**Proof.** If $\mathcal{F}$ is a maximal c-filter and $A \notin \mathcal{F}$, then $\mathcal{F}$ can have no trace on $A$, for such a trace would be a c-filter strictly finer than $A$. Thus $X - A \in \mathcal{F}$. Conversely, if $\mathcal{F}$ is a c-filter which is not maximal, and $\mathcal{G}$ is a strictly finer c-filter, then some c-set $G$ in $\mathcal{G}$ has the property that neither $G$ nor $X - G$ is in $\mathcal{F}$, contrary to the stated condition. $\blacksquare$

A space $X$ is $T_1$-ordered [4] if, for each $z \in X, i(z)$ and $d(z)$ are both closed sets. Note that in a $T_1$-ordered space, each singleton $\{z\}$ is a c-set. A space with closed order is defined to be $T_2$-ordered [4]. A space $X$ is normally ordered [5] if, whenever $A$ and $B$ are disjoint closed sets, with $A$ increasing and $B$ decreasing, there are disjoint open sets $U$ and $V$, with $U$ increasing and $V$ decreasing, such that $A \subseteq U$ and $B \subseteq V$. A space which is both normally ordered and $T_1$-ordered is said to be $T_4$-ordered [3]. Priestly, [6], defined a $C$-space to be one in which $i(A)$ and $d(A)$ are closed whenever $A$ is closed. We define a c-space to be one in which every c-set $A$ has the property that $i(A)$ and $d(A)$ are closed sets. Obviously, every C-space is a c-space; in particular, the compact, $T_2$-ordered spaces are c-spaces. An alternate characterization for c-spaces is given in the next proposition (see [3]).

**Proposition 1.3** If $X$ is a c-space and $A, B$ are c-sets, then $I(A) \cap B = \emptyset$ implies $I(A) \cap D(B) = \emptyset$, and $D(A) \cap B = \emptyset$ implies $D(A) \cap I(B) = \emptyset$. If $X$ is $T_4$-ordered and the two preceding implications hold for arbitrary c-sets $A$ and $B$, then $X$ is a c-space.
The Wallman ordered compactification can be constructed for any $T_1$-ordered space $X$. The original construction by Choe and Park [1] was based on "maximal bifilters"; we shall follow the approach of [3] in which maximal c-filters form the underlying set for $\omega_0 X$. Given a $T_1$-ordered space $X$, let $\omega_0 X = \{ \bar{z} : z \in X \} \cup X'$, where $X'$ is the set of all non-convergent maximal c-filters. A partial order relation is defined for $\omega_0 X$ as follows: $\mathcal{F} \subseteq \mathcal{G}$ iff $I(\mathcal{F}) \subseteq I(\mathcal{G})$ and $D(\mathcal{G}) \subseteq D(\mathcal{F})$. The embedding map $\varphi : X \to \omega_0 X$ given by $\varphi(z) = \bar{z}$ for all $z$ in $X$ is obviously increasing.

For any subset $A$ of $X$, let $A^* = \{ \mathcal{F} \in \omega_0 X : A \in \mathcal{F} \}$. If $\mathcal{F}$ is a filter on $X$, let $\mathcal{F}^*$ be the filter on $\omega_0 X$ generated by $\{ F^* : F \in \mathcal{F} \}$. The fact that the latter collection is a filter base and other important properties of this set operator follow from the next proposition.

**Proposition 1.4** Let $X$ be a $T_1$-ordered space.

(a) For all subsets $A, B$ of $X$, $(A \cap B)^* = A^* \cap B^*$

(b) If $A, B \in C_X$, then $((X - A) \cup (X - B))^* = (X - A)^* \cup (X - B)^*$

(c) If $A \in C_X$, then $(X - A)^* = X^* - A^*$.

**Proof.** Statement (a) is clear, and (b) follows from Proposition 1.2; (c) is an easy consequence of (b).

The topology for $\omega_0 X$ is defined by choosing for a subbase of closed sets the collection $\{ A^* : A \in C_X \}$. If $U \in \mathcal{U}_X$, then $U$ is a finite intersection of complements of c-sets and $U^*$ is open in $\omega_0 X$; indeed $\{ U^* : U \in \mathcal{U}_X \}$ is a base for the open sets in $\omega_0 X$. In particular, sets of the form $V^*$ where $V$ is open and monotone in $X$ form an open subbase for $\omega_0 X$. It should be noted that if $V$ is a non-fundamental open set in $X$, it is not generally true that $V^*$ is open in $\omega_0 X$. The following facts about the topology of $\omega_0 X$ will be stated for future reference.

**Proposition 1.5** Let $X$ be a $T_1$-ordered space.

(a) If $B$ is a monotone closed (respectively, open) set in $X$, then $B^*$ is monotone in the same sense and closed (respectively, open) in $\omega_0 X$.

(b) If $\mathcal{F} \in \omega_0 X$, then the neighborhood filter $\mathcal{V}^*(\mathcal{F})$ at $\mathcal{F}$ in $\omega_0 X$ has for its filter base $\{ U^* : U \in \mathcal{F} \cap \mathcal{U}_X \}$.

The next two theorems summarize the main results already known about the Wallman ordered compactification. Proofs for these propositions form the main results of [1] and [3] and the reader is referred to
these sources for further details. Here we should mention again that the proofs in [1] are formulated in the
language of "bifilters", but the translation into "c-filter" terminology presents no difficulties.

**Theorem 1.6** For any $T_1$-ordered space $X$, $(w_X, \varphi)$ is an ordered compactification of $X$, and $w_X$ is a $T_1$ topological space. Also, $w_X$ is $T_2$-ordered iff $X$ is a $T_4$-ordered $c$-space.

**Theorem 1.7** Let $X$ be a $T_1$-ordered space, $Y$ a $T_2$-ordered compact space, and $f : X \to Y$ a continuous increasing function. Then there is a unique continuous increasing function $f : w_X \to Y$ such that $f \circ \varphi = f$.

Let us recall that for a space which admits a $T_2$-ordered compactification (see [5] for a characterization of such spaces) there is always a largest $T_2$-ordered compactification called the Stone-Čech ordered (or Nachbin) compactification denoted by $\beta_X$ (see [2], [5]). The two preceding theorems yield the following important corollary.

**Corollary 1.8** For a space $X$ which admits a $T_2$-ordered compactification, $w_X$ and $\beta_X$ are equivalent iff $X$ is a $T_4$-ordered $c$-space.

2. Separation Properties of $w_X$.

Given a $T_1$-ordered space $X$, we already know that $w_X$ is $T_1$, and that $w_X$ is $T_2$-ordered iff $X$ is a $T_4$-ordered $c$-space. We shall now examine conditions on $X$ subject to which $w_X$ is $T_1$-ordered or $T_2$. As it turns out, $w_X$ can fail to have either of these latter properties, can have either one without the other, or can have both properties and still fail to be $T_2$-ordered; examples are given later to illustrate all of these possibilities. We begin by finding conditions on $X$ which are necessary and sufficient for $w_X$ to be $T_2$.

**Proposition 2.1** Let $\mathcal{I}$ be an ultrafilter and $\mathcal{G}$ a maximal $c$-filter on $X$. Then $\varphi(\mathcal{I}) \to \mathcal{G}$ in $w_X$ iff $\mathcal{I}^\wedge \subseteq \mathcal{G}$.

**Proof.** Let $\varphi(\mathcal{I}) \to \mathcal{G}$ in $w_X$. Let $F \in \mathcal{I}$ be a $c$-set. If $F \notin \mathcal{G}$ then either $I(F) \notin \mathcal{G}$ or $D(F) \notin \mathcal{G}$; without loss of generality, assume the former. Then $I(F) \notin \mathcal{G}$ implies, by Proposition 1.2, that $X - I(F) \in \mathcal{G}$, and therefore $\mathcal{G} \in (X - I(F))^*$, which is a subbasic open neighborhood of $\mathcal{G}$ in $w_X$. Now $\varphi(\mathcal{I}) \to \mathcal{G}$ implies $(X - I(F))^* \in \varphi(\mathcal{I})$, and consequently $X - I(F) \in \mathcal{I}$. This contradicts the fact that $F \in \mathcal{I}$, and therefore every element of $\mathcal{I}^\wedge$ is in $\mathcal{G}$. 

Conversely, let $\mathcal{F}^\wedge \subseteq \mathcal{G}$, and $(X - A)^*$ be a subbasic open neighborhood of $\mathcal{G}$, where $A$ is closed and monotone in $X$. Since $\mathcal{F}$ is an ultrafilter, either $A \in \mathcal{F}$ or $X - A \in \mathcal{F}$. If $A \in \mathcal{F}$ then $A \in \mathcal{F}^\wedge$, which in turn implies $A \in \mathcal{G}$, contrary to the fact that $X - A \in \mathcal{G}$. Thus $X - A \in \mathcal{F}$, and hence $(X - A)^* \in \varphi(\mathcal{F})$. Since $(X - A)^*$ is an arbitrary subbasic open neighborhood of $\mathcal{G}$, $\varphi(\mathcal{F}) \to \mathcal{G}$. $\blacksquare$

**Theorem 2.2** Let $X$ be a $T_1$-ordered space. Then $\omega_0X$ is $T_3$ iff, for each ultrafilter $\mathcal{F}$ on $X$, there is a unique maximal $c$-filter $\mathcal{G}$ on $X$ such that $\mathcal{F}^\wedge \subseteq \mathcal{G}$.

**Proof.** If $\omega_0X$ is $T_2$ and $\mathcal{F}$ an ultrafilter on $X$, then $\varphi(\mathcal{F})$ is an ultrafilter on $\omega_0X$ which must converge to some maximal $c$-filter $\mathcal{G}$, since $\omega_0X$ is compact. By Proposition 2.1, $\mathcal{F}^\wedge \subseteq \mathcal{G}$. If there is a different maximal $c$-filter $\mathcal{H}$ with $\mathcal{F}^\wedge \subseteq \mathcal{H}$, then $\varphi(\mathcal{F})$ would also converge to $\mathcal{H}$, contrary to the assumption that $\omega_0X$ is $T_2$. Thus $\mathcal{G}$ is unique.

Conversely, assume the uniqueness condition. If $\omega_0X$ is not $T_2$, there is a filter $\mathcal{A}$ on $\omega_0X$ converging to distinct elements $\mathcal{G}, \mathcal{H}$ in $\omega_0X$. Let $\mathcal{F}$ be an ultrafilter on $X$ finer than the filter generated by $\{A \subseteq X : A^* \in \mathcal{A}\}$. One easily verifies that $\varphi(\mathcal{F})$ must converge to both $\mathcal{G}$ and $\mathcal{H}$, which, by Proposition 2.1, violates our assumed uniqueness condition. $\blacksquare$

A space $X$ is defined to be *c-normally ordered* if, for each pair of disjoint $c$-sets $A, B$, there are disjoint fundamental open sets $U, V$ such that $A \subseteq U$ and $B \subseteq V$. As we shall see in later examples, there are spaces which are c-normally ordered but not normally ordered, and vice versa. Of course, both of these versions of “ordered normality” reduce to ordinary normality when the partial order for $X$ is equality.

**Theorem 2.3** The following conditions on a $T_1$-ordered space $X$ are equivalent.

1. $X$ is c-normally ordered.
2. Two disjoint fundamental closed sets in $X$ can be separated by disjoint fundamental open neighborhoods.
3. If $A$ is a $c$-set in $X$, then every fundamental open set containing $A$ contains a fundamental closed set which in turn contains a fundamental open neighborhood of $A$.
4. For each ultrafilter $\mathcal{F}$ on $X$, there is a unique maximal $c$-filter $\mathcal{G}$ finer than $\mathcal{F}^\wedge$.
5. $\omega_0X$ is $T_2$.

**Proof.** The equivalence of (1), (2), and (3) is a routine exercise, and the equivalence of (4) and (5) was established in the previous theorem.
(1) ⇒ (5). If \( \mathcal{F} \) and \( \mathcal{G} \) are distinct maximal \( c \)-filters on \( X \), then there are disjoint \( c \)-sets \( F \) in \( \mathcal{F} \) and \( G \in \mathcal{G} \). Let \( U \) and \( V \) be disjoint fundamental open neighborhoods of \( F \) and \( G \) respectively; then by Proposition 1.4, \( U^* \) and \( V^* \) are disjoint open neighborhoods of \( \mathcal{F} \) and \( \mathcal{G} \), respectively, in \( w_0X \).

(5) ⇒ (1). Let \( A \) and \( B \) be disjoint \( c \)-sets in \( X \). Then \( A^* \) and \( B^* \) are disjoint closed sets in \( w_0X \), and since \( w_0X \) is compact and \( T_2 \), there are disjoint open sets \( M \) and \( N \) in \( w_0X \) such that \( A^* \subseteq M \) and \( B^* \subseteq N \). Since \( \{U^* : U \in \mathcal{U}_X\} \) forms an open base for \( w_0X \), there are subcollections \( \{U_i^* : i \in I\} \) and \( \{V_j^* : j \in J\} \) such that \( M = \bigcup\{U_i^* : i \in I\} \) and \( N = \bigcup\{V_j^* : j \in J\} \). Using the fact that \( A^* \) and \( B^* \) are compact subsets in \( w_0X \), we can find finite subcovers \( U_{i_1}^*, \ldots, U_{i_n}^* \) of \( A^* \) and \( V_{j_1}^*, \ldots, V_{j_m}^* \) of \( B^* \). Letting \( U = U_{i_1} \cup \cdots \cup U_{i_n} \) and \( V = V_{j_1} \cup \cdots \cup V_{j_m} \), we obtain disjoint fundamental open neighborhoods of \( A \) and \( B \) in \( X \).

Although neither of the properties "normally ordered" and "\( c \)-normally ordered" implies the other in general, the next theorem establishes the equivalence of these properties in \( T_1 \)-ordered \( c \)-spaces. We first need the following lemma.

**Lemma 2.4** Let \( X \) be a \( c \)-normally ordered \( c \)-space. If \( A \) is a \( c \)-set in \( X \) and \( U \) is an open, increasing neighborhood of \( A \), then there is a closed, increasing neighborhood \( G \) of \( A \) such that \( A \subseteq G \subseteq U \).

**Proof.** Let \( B = X - U \); by Proposition 1.3, \( I(A) \cap B = \emptyset \), and so \( I(A) \) and \( B \) can be separated by disjoint, fundamental open sets \( W \) and \( V \), respectively. By Proposition 1.1, \( X - V \) is a finite union of \( c \)-sets \( C_1, \ldots, C_n \). By Proposition 1.3, \( I(C_i) \cap B = \emptyset \) for all indices \( i \); let \( G = \bigcup\{I(C_i) : i = 1, \ldots, n\} \). Thus \( G \) is closed and increasing, and \( A \subseteq W \subseteq G \subseteq U \).

**Theorem 2.5** For a \( T_1 \)-ordered \( c \)-space \( X \), the following statements are equivalent.

(a) \( X \) is normally ordered.

(b) \( X \) is \( c \)-normally ordered.

(c) \( w_0X \) is \( T_2 \)-ordered.

**Proof.** (a) ⇔ (c) is established in Theorem 1.6. (c) ⇒ (b) follows by Theorem 2.3. (b) ⇒ (c): It is sufficient to show that if \( \mathcal{F}, \mathcal{G} \in w_0X \) and \( \mathcal{F} \not\supseteq \mathcal{G} \), then there are disjoint neighborhoods of \( \mathcal{F} \) and \( \mathcal{G} \) in \( w_0X \), where the former is an increasing set and the latter decreasing. If \( \mathcal{F} \not\supseteq \mathcal{G} \), then either \( I(\mathcal{F}) \not\subseteq \mathcal{G} \) or \( D(\mathcal{G}) \not\subseteq \mathcal{F} \); without loss of generality, assume the latter. Since \( \mathcal{F} \) is a maximal \( c \)-filter, there are \( c \)-sets \( G \in \mathcal{G} \) and \( F \in \mathcal{F} \) such that \( D(G) \cap F = \emptyset \). By Lemma 2.4, there is a closed, increasing neighborhood \( N \) of \( F \) such that \( N \cap D(G) = \emptyset \) and a fundamental open set \( W \) such that \( F \subseteq W \subseteq N \). Now \( (X - N)^* \), which is
a decreasing open set in \( w_0X \) by Proposition 1.5, and the increasing hull of \( W^* \) in \( w_0X \) provide the desired neighborhoods which separate \( \mathcal{F} \) and \( \mathcal{G} \) in \( w_0X \).

If \( w_0X \) is \( T_2 \)-ordered, then \( X \) is necessarily a \( T_1 \)-ordered c-space; thus the following corollary is immediate.

**Corollary 2.6** For a \( T_1 \)-ordered space \( X \), \( w_0X \) is \( T_2 \)-ordered iff \( X \) is a c-normally ordered c-space.

**Theorem 2.7** If \( X \) is a \( T_1 \)-ordered c-space, then \( w_0X \) is \( T_1 \)-ordered.

**Proof.** Let \( \mathcal{F} \in w_0X \), and let \( i_*(\mathcal{F}) = \{ G \in w_0X : \mathcal{F} \preceq G \} \) be the increasing hull of \( \mathcal{F} \) in \( w_0X \). Let \( \text{"cl}_w \) denote the closure operator in \( w_0X \). We shall show that \( \text{cl}_w(i_*(\mathcal{F})) \subseteq i_*(\mathcal{F}) \); a similar argument shows that the decreasing hull of \( \mathcal{F} \) is closed, and hence that \( w_0X \) is \( T_1 \)-ordered.

If \( G \in \text{cl}_w(i_*(\mathcal{F})) \), then for each \( A \in C_X \) such that \( G \in (X-A)^* \), there is \( H \in i_*(\mathcal{F}) \) such that \( H \in (X-A)^* \). With the help of Proposition 1.2, the last sentence may be restated as follows: if \( G \in \text{cl}_w(i_*(\mathcal{F})) \) and \( A \in C_X \), then \( A \not\subseteq G \) implies there is \( H \in w_0X \) such that \( \mathcal{F} \preceq H \) and \( A \not\subseteq H \).

Now assume that \( G \not\subseteq \text{cl}_w(i_*(\mathcal{F})) \); if \( G \not\subseteq i_*(\mathcal{F}) \), then either \( I(\mathcal{F}) \not\subseteq G \) or \( D(G) \not\subseteq \mathcal{F} \). Suppose \( I(\mathcal{F}) \not\subseteq G \); then there is \( F \in \mathcal{F} \) such that \( I(F) \not\subseteq G \). But \( G \in \text{cl}_w(i_*(\mathcal{F})) \) implies there is \( H \in i_*(\mathcal{F}) \) such that \( I(F) \not\subseteq H \), a contradiction. On the other hand, suppose \( D(G) \not\subseteq \mathcal{F} \). Since \( \mathcal{F} \) is a maximal c-filter, there are c-sets \( F \in \mathcal{F} \) and \( G \in \mathcal{G} \) such that \( D(G) \cap F = \phi \), and by Proposition 1.3, \( D(G) \cap I(F) = \phi \). Thus \( I(F) \not\subseteq \mathcal{G} \), and a repetition of the preceding argument again yields a contradiction. We may therefore conclude that \( G \in i_*(\mathcal{F}) \), and hence that \( i_*(\mathcal{F}) \) is closed.

The converse of Theorem 2.7 does not hold in general, however the next theorem establishes a partial converse. We shall say that a net \( (x_\lambda)_{\lambda \in \Lambda} \) in a space \( X \) is **upward directed** if, for each pair of indices \( \lambda, \mu \in \Lambda \), there is \( \sigma \in \Lambda \) such that \( \lambda \leq \sigma, \mu \leq \sigma, x_\lambda \leq x_\sigma \), and \( x_\mu \leq x_\sigma \). **Downward directed** nets are defined dually.

**Theorem 2.8** Let \( X \) be a \( T_2 \)-ordered space with the property that, whenever \( A \) is decreasing (respectively increasing) and \( z \in cl_X A \), there is an upward directed (respectively, downward directed) net on \( A \) which converges to \( z \). Then \( w_0X \) is \( T_1 \)-ordered iff \( X \) is a c-space.

**Proof.** Suppose \( X \) is not a c-space. Then for some c-set \( A \) in \( X \), either \( i(A) \) is not closed or else \( d(A) \) is not closed. There is no loss of generality in assuming the latter. Thus there is some \( y \in cl_X d(A) \) such that \( y \not\in d(A) \); by assumption there is an upward directed net \( (x_\lambda)_{\lambda \in \Lambda} \) on \( d(A) \) converging to \( y \). Then \( \{ s(z_\lambda) \cap A : \lambda \in \Lambda \} \) is a base for a c-filter \( \mathcal{G} \) on \( X \); let \( \mathcal{F} \) be a maximal c-filter finer than \( \mathcal{G} \).
Let $\mathcal{K}$ be the filter generated by the net $(z_\lambda)_{\lambda \in A}$. Then $\mathcal{K} \rightarrow y$, and if $\mathcal{F} \rightarrow z$ for some $z \in X$, then it must follow that $y \leq z$, since $\mathcal{K} \times \mathcal{F}$ has a trace on the order, and the order is closed. But $y \leq z$ is a contradiction since $A \in \mathcal{F}$ and $A$ a c-set implies $z \in A$, and therefore $y \in d(A)$. Thus $\mathcal{F}$ must be a non-convergent maximal c-filter, and therefore an element of $w_0 X$.

One may easily verify that $z = A \otimes Y$ holds for all $\lambda \in A$, but that $y \nsubseteq \mathcal{F}$. But $(z_\lambda)_{\lambda \in A} \rightarrow y$ in $w_0 X$, and hence $y \in cl_{w_0}(d(A))$, but $y \nsubseteq d(A)$. Thus $w_0 X$ is not $T_1$-ordered. \*[1]

**Example 2.9** Let $X = A \cup B \cup \{a\} \cup \{b\}$, where $A = \{z_i : i = 1, 2, 3, \cdots\}$ and $B = \{y_i : i = 1, 2, 3, \cdots\}$. Define the topology for $X$ by specifying that $\{z\}$ is open for $z \in A \cup B$; the neighborhood filter at $a$ (respectively, $b$) is generated by sets of the form $A_n = \{z_i : i \geq n\}$ (respectively, $B_n = \{y_i : i \geq n\}$), where $n = 1, 2, 3, \cdots$. The order for $X$ is the smallest partial order such that $z_i \leq y_i$ for each positive integer $i$.

It is evident from this construction that $X$ is a compact, $T_2$, $T_1$-ordered space; thus we may identify $X$ with $w_0 X$. Note that every closed set in $X$ is a c-set, and every open set is a fundamental open set; it follows that $X$ is c-normally ordered. However $X$ is not a c-space, since for the c-set $C = B \cup \{b\}$, $i(C) = A \cup C$ is not closed. Thus $X$ is neither $T_2$-ordered nor normally ordered.

The main points illustrated by this example are that the converse of Theorem 2.7 does not hold in general, and that the conditions $T_1$-ordered, $T_2$, and c-normally ordered on $X$ are not sufficient to guarantee that $w_0 X$ is $T_2$-ordered. This example also shows that a c-normally ordered space need not be normally ordered, even when the axioms $T_1$-ordered and $T_2$ are present.

3. **Examples in Euclidean Space.**

Additional insight into the behavior of the Wallman ordered compactification can be gained by studying some simple examples in $R^n$ (by which we mean Euclidean $n$-space with the usual product topology and product order), especially in the case $n = 2$. We shall show that $n = 2$ is the largest value of $n$ for which $w_0 R^n$ is $T_2$-ordered, and hence the largest value of $n$ for which $w_0 R^n = \beta_0 R^n$. We shall use the known properties $w_0 R^2$ to describe $\beta_0 R^2$. We also examine two simple subspaces of $R^2$ for which the Wallman ordered compactification is not $T_2$-ordered.

**Theorem 3.1** Let $A$ be a closed, convex subset of $R^2$. Then $i(A) = I(A)$ and $d(A) = D(A)$, and hence $R^2$ is a c-space.
Proof. We shall prove that $i(A)$ is closed; a similar argument shows that $d(A)$ is closed.

If $i(A)$ is not closed, then there is a sequence $(z_n, y_n)$ in $i(A)$ such that $(z_n, y_n) \to (z_0, y_0)$, where $(z_0, y_0) \notin i(A)$. Let $(a_n, b_n)$ be a sequence in $A$ such that $(a_n, b_n) \leq (z_n, y_n)$ for all $n \in \mathbb{Z}^+$. The convergence of the sequence $(z_n, y_n)$ implies that the sequences $(a_n)$ and $(b_n)$ are both bounded above. Either of these sequences may fail to be bounded below, and this leads us to consider four cases.

Case 1. $(a_n)$ and $(b_n)$ are both bounded below. Then there is a convergent subsequence $(a_{n_k}, b_{n_k}) \to (a, b)$. Since $A$ is closed, $(a, b) \in A$, and since $R^2$ is $T_2$-ordered, $(a, b) \leq (z_0, y_0)$, contrary to $(z_0, y_0) \notin i(A)$.

Case 2. $(a_n)$ and $(b_n)$ are both unbounded below. Then for some $n \in \mathbb{Z}^+$, $(a_n, b_n) \leq (z_0, y_0)$, which again contradicts $(z_0, y_0) \notin i(A)$.

Case 3. $(a_n)$ is bounded below, but $(b_n)$ is not. In this case, there is no loss of generality in assuming that $a_n \to a$ and that $(b_n)$ is an unbounded, decreasing sequence. Then there is $n_0 \in \mathbb{Z}^+$ such that $b_n \leq y_0$, for all $n \geq n_0$. Also $a \leq z_0$ since $R^4$ is $T_2$-ordered, and for $n \geq n_0$ we must have $a \leq a_n$, for otherwise $(a_n, b_n) \leq (z_0, y_0)$ would again yield a contradiction. Thus from the sequence $(a_n)_{n \geq n_0}$ it is possible to obtain a decreasing subsequence $(a_{n_k}) \to a$, and the corresponding subsequence $(b_{n_k})$ is, of course, decreasing and unbounded. Now for any $j \in \mathbb{Z}^+$, $(a_{n_j}, b_{n_j}) \leq (a_{n_1}, b_{n_1}) \leq (a_{n_1}, b_{n_1})$, and the convexity of $A$ implies that $(a_{n_j}, b_{n_j}) \in A$ for all $j \in \mathbb{Z}^+$. Thus, $(a_{n_j}, b_{n_j}) \to (a, b_{n_1})$ and $(a, b_{n_1}) \in A$ since $A$ is closed. But $a \leq z_0$ and $b_{n_1} \leq y_0$ implies $(z_0, y_0) \in i(A)$, a contradiction.

Case 4. $(b_n)$ is bounded below, but $(a_n)$ is not. An argument similar to that of case 3 yields a contradiction.

Proposition 3.2 $R^n$ is not a c-space for $n \geq 3$.

Proof. Let $A = \{(m, -\frac{1}{m}, \frac{1}{m}, 0, \ldots, 0) \in R^n : m \in \mathbb{Z}^+\}$. The elements of $A$ are isolated in both the topological and order sense, and so $A$ is a closed, convex subset of $R^n$. One can verify that $I(A) = i(A)$ and $D(A) = d(A) \cup B$, where $B = \{(x, 0, z, 0, \ldots, 0) \in R^n : z \leq 0\}$. Since $i(A) \cap B = \emptyset$, it follows that $A = I(A) \cap D(A) = A^\wedge$, and so $A$ is a c-set. But $d(A) \neq D(A)$, and so $R^n$ is not a c-space.

Theorem 3.3 $R^n$ is $T_4$-ordered for all $n \in \mathbb{Z}^+$.

Proof. If $z = (z_1, \ldots, z_n) \in R^n$ and $z \leq y$, then $N(z, \epsilon) \subseteq d(N(y, \epsilon))$ and $N(y, \epsilon) \subseteq i(N(z, \epsilon))$; from this it easily follows that the increasing and decreasing hulls of open sets in $R^n$ are open. If $A$ is a closed increasing set and $B$ a closed, decreasing set in $R^n$ such that $A \cap B = \emptyset$, then for each $b \in B$ we may choose
such that $A \cap N(b, r_b) = \phi$, and consequently $A \cap d(N(b, r_b)) = \phi$. Likewise, for each $a \in A$, there is $r_a$ such that $B \cap i(N(a, r_a)) = \phi$. Let $U = \cap\{i(N(a, r_a/2)) : a \in A\}$ and $V = \cup\{d(N(b, r_b/2)) : b \in B\}$. Then $U$ and $V$ are disjoint open sets, the former increasing and the latter decreasing, which separate $A$ and $B$. 

**Theorem 3.4** The following statements are equivalent.

(a) $\beta^n$ is a $c$-space.

(b) $\omega_0 \mathbb{R}^n$ is $T_1$-ordered.

(c) $\omega_0 \mathbb{R}^n$ is $T_2$-ordered.

(d) $\omega_0 \mathbb{R}^n = \beta_0 \mathbb{R}^n$.

(e) $n \leq 2$.

**Proof.** It is obvious that in $\mathbb{R}^1$, the increasing or decreasing hull of any closed set is closed, and so $\mathbb{R}^1$ is a $c$-space. Thus (a) $\Leftrightarrow$ (c) follows by Theorem 3.1 and Proposition 3.2.

(a) $\Leftrightarrow$ (b) follows from Theorems 2.7 and 2.8. By Theorems 1.6, 3.1, and 3.3, $\omega_0 \mathbb{R}^n$ is $T_2$-ordered for $n \leq 2$, and by Theorem 1.6 and Proposition 3.2, $\omega_0 \mathbb{R}^n$ is not $T_2$-ordered for $n \geq 3$; thus (c) $\Leftrightarrow$ (e). Finally, (c) $\Leftrightarrow$ (d) follows by Theorem 1.6 and Corollary 1.8.

The Wallman ordered compactification of $\mathbb{R}^1$ is the familiar two point compactification which is commonly called the "extended real line." In the case of $\mathbb{R}^2$, this compactification, which is not so familiar, is described in the next example.

**Example 3.5** $\mathbb{R}^2$ is simultaneously homeomorphic and order isomorphic to the open square $S = \{(x_1, x_2) \in \mathbb{R}^2 : -1 < x_1 < 1, -1 < x_2 < 1\}$. The closed square $\bar{S} = \{(x_1, x_2) \in \mathbb{R}^2 : -1 \leq x_1 \leq 1, -1 \leq x_2 \leq 1\}$ can thus be regarded as a $T_2$-ordered compactification of $\mathbb{R}^2$. The most convenient way to describe $\omega_0 \mathbb{R}^2$ (or, equivalently, $\beta_0 \mathbb{R}^2$) is to consider each boundary (i.e., compactification) point $p$ of $S$ to be replaced by the set $\mathcal{M}_p$ of all non convergent maximal $c$ filters $\mathcal{N}$ on $S$ which converge to $p$ in $\bar{S}$. If $p = (1, 1)$, then $\mathcal{M}_p$ consists of a single maximal $c$-filter which is the greatest element of $\omega_0 \mathbb{R}^2$.

Similarly, the least element of $\omega_0 \mathbb{R}^2$ is the unique maximal $c$-filter in $\mathcal{M}_{(-1,-1)}$. If $p$ is any boundary point of $S$ other than $(1, 1)$ or $(-1,-1)$, then $\mathcal{M}_p$ contains $2^\gamma$ distinct elements, where $\gamma$ is the cardinality of the real line, including both a greatest and a least element. For instance, if $p = (1, 0)$ the least element in $\mathcal{M}_p$ is a maximal $c$-filter in $S$ which contains the positive $x$ axis and converges in $\bar{S}$ to $(1, 0)$; the greatest element is a maximal $c$-filter finer than the filter supremum of $\{I(\mathcal{F}) : \mathcal{F} \in \mathcal{M}_p\}$ which converges to $(1, 0)$ in $\bar{S}$.
If \( p, q \) are two boundary points in \( S \) and \( p \leq q \) in \( S \), then \( \mathcal{G} \preceq \mathcal{H} \) for all \( \mathcal{G} \in \mathcal{M}_p \), and for all \( \mathcal{H} \in \mathcal{M}_q \); furthermore if \( \mathcal{G} \preceq \mathcal{H} \) for some \( \mathcal{G} \in \mathcal{M}_p \), and for some \( \mathcal{H} \in \mathcal{M}_q \), then \( p \leq q \) in \( S \). Similarly if \( z \in S \) and \( p \) is a boundary point of \( S \), then \( z \preceq p \) in \( S \) iff \( \mathcal{Z} \preceq \mathcal{H} \) for some \( \mathcal{H} \in \mathcal{M}_p \), (in which case \( \mathcal{Z} \preceq \mathcal{H} \) for all \( \mathcal{H} \in \mathcal{M}_p \)).

The next two examples show that even for the simplest subspaces \( X \) of \( R^2 \), various pathologies can arise in \( w_0 X \).

Example 3.6 Let \( X_1 \) be the closed square \( S \) (defined in Example 3.5) with the origin \((0,0)\) deleted, and with the topology and order inherited from \( R^2 \). Let \( \mathcal{G} \) (respectively, \( \mathcal{H} \)) be the maximal c-filter on \( X_1 \) which contains the negative portion of the x-axis (respectively, y-axis) and converges to \((0,0)\) in \( S \). If \( A = \{ (z,0) : z < 0 \} \) and \( B = \{ (0,y) : y < 0 \} \), then \( A \) and \( B \) are c-sets in \( X_1 \). Since \( B \subseteq D(A) \) but \( B \cap D(A) = \emptyset \), \( d(A) \neq D(A) \) and thus \( X_1 \) is not a c-space. Furthermore, it follows from Theorem 2.8 that \( w_0 X_1 \) is not \( T_1 \)-ordered. Also, \( A \) and \( B \) cannot be separated by fundamental open sets, and consequently \( w_0 X_1 \) is not \( T_2 \). However, the argument used to prove Theorem 3.3 can be applied to show that \( X_1 \) is \( T_4 \)-ordered. We thus have an example which, in contrast to Example 2.9, is normally ordered but not c-normally ordered, and for which the Wallman ordered compactification is neither \( T_1 \)-ordered nor \( T_2 \).

It is easy to describe \( w_0 X_1 \). The "hole" at \((0,0)\) in \( X_1 \) is filled in \( w_0 X_1 \) by a set \( \mathcal{M}_{(0,0)} \) of maximal c-filters on \( X_1 \) which converge to \((0,0)\) in \( S \). The filters \( \mathcal{G} \) and \( \mathcal{H} \) described above are maximal elements in \( \mathcal{M}_{(0,0)} \); there are also two maximal elements in \( \mathcal{M}_{(0,0)} \) which are maximal c-filters converging to \((0,0)\) in \( S \) along the positive \( z \) and \( y \) axes. The set of compactification points contains no greatest or least element and has cardinality \( 2^\omega \).

Example 3.7 Let \( X_2 \) be the closed square \( S \) with the \( y \) axis deleted except for the origin; the topology and order are those inherited from \( R^2 \). One may show that this space is both \( T_4 \)-ordered and c-normally ordered. However, \( X_2 \) is not a c-space, for if \( A = \{ (z, \frac{1}{z}) : z < 0 \} \), then \( A \) is a c-set and \((0,0) \in D(A) \). Thus, \( w_0 X_2 \) is \( T_2 \) by Theorem 2.3, but \( w_0 X_2 \) is not \( T_1 \)-ordered by Theorem 2.8. Without going into detail, we can partially describe \( w_0 X_2 \) by remarking that every "hole" in \( X_2 \) corresponding to a missing point on the \( y \)-axis is filled in \( w_0 X_2 \) by adding \( 2^\omega \) compactification points including, in each case, a pair of minimal elements and a pair of maximal elements.

For the sake of completeness, we should give an example of a space \( X \) for which \( w_0 X \) is \( T_1 \)-ordered but not \( T_2 \). This turns out to be quite easy. Let \( X \) be any \( T_2 \), completely regular topological space which is
not normal, and let the order for $X$ be equality. Then it is well known that $w_0X$ is $T_1$ (and hence $T_1$-ordered) but not $T_2$.

Our final example does not pertain directly to the Wallman ordered compactification, but it does provide further insight into the nature of $c$-sets, which are crucial ingredients in the construction of this compactification. It shows that closed, convex subsets of $R^3$ need not be $c$-sets, and that the relation defined by "$A$ is a $c$-set in $B$" is not transitive. We are grateful to Dr. Bettina Zoeller for providing this example as well as the related example used in the proof of Proposition 3.2.

Example 3.8 Let $K = \{(m, -\frac{1}{m}, \frac{1}{m}): m \in Z^+\} \cup \{(-m, \frac{1}{m}, -\frac{1}{m}): m \in Z^+\}$ be a subset of $R^3$; note that $K$ is closed and convex. Let $L = I(K) \cap D(K)$; then $L$ is the union of $K$ with the $x$-axis, and consequently $K$ is not a $c$-set. Furthermore, observe that $K$ is a $c$-set in $L$ (considered as a subspace of $R^3$) and $L$ is a $c$-set in $R^3$, but $K$ is not a $c$-set in $R^3$.

Such an example cannot be found in $R^n$ for $n \leq 2$, since in these spaces every closed, convex set is a $c$-set.

References


