ON $\alpha$-CONVEX FUNCTIONS OF ORDER $\beta$ WITH $m$-FOLD SYMMETRY

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ABSTRACT. This note is a continuation of the previous work [1,2,3]. First we get a new subordination for $\alpha$-convex functions of order $\beta$ when $\alpha=1-2\beta$, which implies the rotation theorem for $(1-2\beta)/m$-convex functions of order $\beta$ with $m$-fold symmetry. Then we extend the known results on $\alpha$-convex functions of order $\beta$ to the functions with $m$-fold symmetry. In particular, we give the sharp order of convexity of $\alpha$-convex functions of order $\beta$ with $m$-fold symmetry for $\alpha>1$, which is analogous in sharpness to a result given by Miller, Mocanu and Reade [1].

KEYWORDS AND PHRASES. Subordination, $\alpha$-convex functions of order $\beta$, $m$-fold symmetry, rotation theorem, order of convexity, distortion theorems.

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1. INTRODUCTION.

Let $J_m(\alpha,\beta)$ be the class of $\alpha$-convex functions of order $\beta$ with $m$-fold symmetry, where $\alpha>0$, $0<\beta<1$ and $m=1,2,\ldots$. That is, it consists of analytic functions $f(z)=z+\sum_{n=1}^{\infty}a_{nm}z^{nm+1}$ in the unit disk $D=\{z:|z|<1\}$ with $f(z)f'(z)/z\neq0$ and

$$\text{Re}\{(1-\alpha)zf'(z)/f(z)+\alpha(1+zf''(z)/f'(z))\} \geq \beta. \quad (1.1)$$

In [1], Miller, Mocanu and Reade studied the class $J(\alpha,0)=J_{1}(\alpha,0)$. Liu [2] and we [3] discussed the class $J(\alpha,\beta)=J_{1}(\alpha,\beta)$. Liu got the sharp bounds of $|f(z)|$, $|a_3^3-ua_2^2|$ ($-\infty<\mu<\infty$) and $|\arg f'(z)|$ for $\alpha=0,1$. In [3], we obtained a subordination result for $J(\alpha,\beta)$, some distortion theorems, etc.

This note is a continuation of previous work. First we get a new subordination theorem for the class $J(1-2\beta,\beta)$, which implies the rotation theorem for $J_m((1-2\beta)/m,\beta)$. Then we extend known results on $J(\alpha,\beta)$ to the class $J_m(\alpha,\beta)$. In particular, we give the sharp order of convexity of functions in the class $J_m(\alpha,\beta)$ for $\alpha>1$, which is analogous in sharpness to a result given by Miller, Mocanu and Reade [1].
2. SUBORDINATION AND DISTORTION PROPERTIES.

At first, we establish a homeomorphic relation between $J_m(\alpha, \beta)$ and $J_m(\alpha, \beta)$.

**Lemma 1.** $f(z) \in J_m(\alpha, \beta)$ if and only if $g(z) \in J_m(\alpha, \beta)$, where $g(z) = f(z^{1/m})$.

**Proof.** If $f(z) \in J_m(\alpha, \beta)$, then $g(z)$ is also analytic in $D$. It is not difficult to show that $g(z)g'(z)/z \neq 0$ and

$$
(1-\alpha)zf'(z)/f(z) + \alpha(1+zf''(z)/f'(z))
$$

in $D$, where $u = z^{1/m}$. Hence $g(z) \in J_m(\alpha, \beta)$. Similarly, we can prove

$$
f(z) = g(z^{1/m}) \in J_m(\alpha, \beta) \text{ if } g(z) \in J_m(\alpha, \beta).$$

This completes the proof.

It is well known that $G(z) \in J_m(0, \beta)$ if and only if there is a probability measure $\mu(x)$ on the unit circle $X = \{x: |x| = 1\}$ such that

$$G(z) = z \exp\{2(1-\beta) \int X - \log(1-xz) d\mu(x)\}.
$$

This implies, by Lemma 1, that $F(z) \in J_m(0, \beta)$ if and only if there is a probability measure $\mu(x)$ on $X$ such that

$$F(z) = z \exp\{2(1-\beta) \int X - \log(1-xz^m) d\mu(x)\}.
$$

(2.2)

Because $g(z) \in J_m(\alpha, \beta)$ if and only if there is a $G(z) \in J_m(0, \beta)$ such that [2]

$$g(z) = \{a-1 \int_0^1 u - 1G(u) 1/m du\}^m,
$$

we have for $\alpha > 0$ that $f(z) \in J_m(\alpha, \beta)$ if and only if there is a $F(z) \in J_m(0, \beta)$ such that

$$f(z) = (a-1 \int_0^1 u - 1F(u) 1/\alpha du)^\alpha.
$$

(2.3)

From (2.2) and (2.3), we obtain the following result.

If $f(z) \in J_m(\alpha, \beta)$ and $|z| < 1$, then

$$e^{-1m^{1/m} k_m(\alpha, \beta, re^{im}/m) \leq |f(z)| \leq k_m(\alpha, \beta, r)}.
$$

(2.4)

where

$$k_m(\alpha, \beta, z) = \begin{cases} 
    z(1-z^m)^{-2(1-\beta)/m} & (\alpha = 0) \\
    a^{-1} \int_0^1 u - 1a(1-u^m)^{-2(1-\beta)/\alpha m} du & (\alpha > 0) 
\end{cases}
$$

(2.5)

is the $\alpha$-convex Koebe function of order $\beta$ with $m$-fold symmetry. Specifically, we denote $k_1(\alpha, \beta, z)$ by $k(\alpha, \beta, z)$.

In order to state our subordination theorem, we shall make use of the following lemma.
LEMMA 2. Let \( \log q(z) \) be a convex univalent function in \( D \) and 
\[
 p_i(z) \ll q(z) \quad (i=1,2,\ldots,n). 
\]
Then for \( \lambda_i > 0 \) and \( \sum_{i=1}^{n} \lambda_i = 1 \), 
\[
 \prod_{i=1}^{n} p_i(z)^{\lambda_i} \ll q(z). 
\]

PROOF. Since \( \log q(z) \) is a convex function and \( p_i(z) \ll q(z) \), we have 
\( p_i(z) \neq 0 \) and \( \log p_i(z) \ll \log q(z) \), which implies 
\[
 \log p_i(D) \ll \log q(D). 
\]
From the fact that \( \log q(D) \) is a convex domain, we get for each \( z \in D \)
\[
 \sum_{i=1}^{n} \lambda_i \log p_i(z) \ll \log q(z), 
\]
and then 
\[
 \sum_{i=1}^{n} \lambda_i = 1 \log p_i(z) \ll \log q(z), 
\]
which is equivalent to the desired result.

COROLLARY 1. If \( p_i(z) \ll (1-bz)/(1-az) \) \( (i=1,2,\ldots,n, -1<a,b<1) \), then for 
\( \lambda_i > 0 \) and \( \sum_{i=1}^{n} \lambda_i = 1 \) we have 
\[
 \prod_{i=1}^{n} p_i(z)^{\lambda_i} \ll (1-bz)/(1-az). 
\]

PROOF. For \( a=b \), the result is trivial. For \( a \neq b \), we know \( \log(1-bz) - \log(1-az) \) 
is a convex function. Hence the required result follows from Lemma 2. 
This corollary and some of its applications may be found elsewhere [4].

THEOREM 1. Let \( g(z) \in \mathcal{K}(1-2\beta, \beta) \) and \( 0<\lambda<1 \), then 
\[
 g'(z)^{\lambda}(g(z)/z)^{1-2\lambda} < 1/(1-z). 
\]
In particular, we have 
\[
 g'(z) < 1/(1-z)^2, \quad (2.7) 
\]
\[
 g(z)/z < 1/(1-z). \quad (2.8) 
\]

PROOF. First we prove (2.8). 
If \( \beta = \frac{1}{2} \), then (1.1) becomes \( \Re\{zg'(z)/g(z)\} > \frac{1}{2} \), which gives 
\[
 zg'(z)/g(z) < 1/(1-z). 
\]
If \( \beta < \frac{1}{2} \), we know [3] 
\[
 zg'(z)/g(z) < zk'((1-2\beta, \beta, z)/k(1-2\beta, \beta, z)=1/(1-z)). 
\]
In both of these cases, we have

\[
\frac{zg'(z)}{g(z)} - 1 \leq \frac{z}{1-z}.
\]

Since \( \frac{z}{1-z} \) is convex [5],

\[
\int_0^1 \frac{z}{1-u} \frac{1}{g(u)} du = \int_0^1 \frac{z}{1-u} du.
\]

That is, \( \log g(z) = \log \frac{1}{1-z} \), which is equivalent to (2.8).

By using Corollary 1 for \( p_1(z) = \frac{zg'(z)}{g(z)} \), \( p_2(z) = g(z)/z \), \( \lambda_1 = \lambda \), and \( \lambda_2 = 1-\lambda \), we obtain (2.6). The proof is completed.

**THEOREM 2.** Let \( f(z) \in J_{m}(1-2\beta)/m, 0 < \alpha < 1 \) and \( |z| = r < 1 \), then we have the sharp estimates

\[
1/(1-r^m) \leq f'(z) \leq \frac{m(1-\lambda) - \gamma}{1-r^m},
\]

\[
\frac{\lambda}{1-\gamma} \leq \frac{\arg f(z)}{z} - \frac{\lambda}{1-\gamma} \leq \frac{\lambda}{1-\gamma}.
\]

**PROOF.** Let \( g(z) = f(z^{1/m})^m \), we know \( g(z) \in J_{m}(1-2\beta)/m, 0 < \alpha < 1 \) from Lemma 1 and \( zg'(z)/f(z) = ug'(u)/g(u) \), where \( u = z^m \). Let

\[
p(z) = g'(z)^{\lambda}(g(z)/z)^{1-2\lambda}, p_1(z) = f'(z)^{\lambda}(f(z)/z)^{(1-\lambda)m-\lambda},
\]

then

\[
p_1(z) = (zf'(z)/f(z))^{\lambda}(f(z)/z)^{(1-\lambda)m} = (ug'(u)/g(u))^{\lambda}(g(u)/u)^{1-\lambda} = p(u).
\]

From Theorem 1 and the principle of subordination, we have

\[
p(z) \in J_{m}(1-2\beta)/m, 0 < \alpha < 1 \), where \( q(z) = 1/(1-z) \). This implies \( p_1(z) \in J_{m}(1-2\beta)/m, 0 < \alpha < 1 \), where \( q_1(z) = 1/(1-z^m) \), which gives the results. This completes the proof of theorem 2.

The inequality (2.10) contains the following rotation theorem for \( J_{m}(1-2\beta)/m, \beta \).

**COROLLARY 2.** If \( f(z) \in J_{m}(1-2\beta)/m, 0 < \alpha < 1 \), then

\[
|\arg f'(z)| < (m+1)\arcsin r^m/m.
\]

The following subordination is due to Liu [2].

\[
g'(z)^{\alpha}(g(z)/z)^{1-\alpha} (1-z)^{-2(1-\beta)} \leq (1-\beta)
\]

whenever \( g(z) \in J(\alpha, \beta) \). In [3] we found that if \( g(z) \in J(\alpha, \beta) \), then

\[
zg'(z)/g(z) \leq zk'(\alpha, \beta, z)/k(\alpha, \beta, z).
\]
By using a method similar to that used in the proof of theorem 2, we can obtain the following theorems from (2.12) and (2.13). We omit most of their proofs.

When \( m=1 \), most of the following results were given in [2] and [3] respectively.

**THEOREM 3.** Let \( f(z) \in J(\alpha, \beta) \), \( |z|=r<1 \), then we have the sharp results

\[
\begin{align*}
r^{1-\alpha}/(1+r^m)^2(1-\beta)/m &< |f'(z)|^\alpha |f(z)|^{1-\alpha}(1-r^m)^2(1-\beta)/m, \\
|\arg[f'(z)\alpha(f(z)/z)^{1-\alpha}]| &< 2(1-\beta)m^{-1}\arcsin m, \\
\text{Re}[f'(z)\alpha(f(z)/z)^{1-\alpha}] &> 2(1-\beta)/m.
\end{align*}
\]

**THEOREM 4.** Let \( f(z) \in J(\alpha, \beta) \), \( |z|=r<1 \), then we have the sharp inequalities

\[
\begin{align*}
\text{Re}[\pi/k_m(\alpha, \beta, \text{Re}1/m)] &< |zf'(z)/f(z)| < \text{Re}[\pi/k_m(\alpha, \beta, \text{Re}1/m)], \\
|\arg(zf'(z)/f(z))| &< \text{Re}[\arg(zk_m(\alpha, \beta, z)/k_m(\alpha, \beta, z))].
\end{align*}
\]

**PROOF.** We give an outline of the proof of (2.17). Let

\[
p(z) = \frac{zf'(z)}{f(z)}, \quad q(z) = \frac{zk'(\alpha, \beta, z)}{k(\alpha, \beta, z)}.
\]

We prove \( \max |q(z)| = q(\cdot) \) and \( \min |q(z)| = q(\text{Re}1/m) \).

Let \( q(z) = 1 + B_1 z + B_2 z^2 + \ldots \), it follows from

\[
q(z) + \alpha q'(z)/q(z) = (1+(1-2\beta)z^m)/(1-z^m)
\]

that

\[
(1+\alpha n)B_n = 2(1-\beta) + \sum_{k=1}^{n-1} (2-2\beta - n-k)B_k.
\]

By using the fact that \( \text{Re}(q(z)) > \beta [3] \), we have \( |B_k| < 2(1-\beta) [6] \). Hence we get \( B_n > 0 \) \( (n=1,2,\ldots) \) by induction and also \( \max |q(z)| = q(r) \).
Because the coefficients of \( q(z) \) are all real and \( q(z) \) is \( m \)-fold symmetric, we can obtain
\[
\min_{|z|=r} |q(z)| = q(re^{i\pi/m}) \text{ by proving}
\]
\[
|q(re^{i\theta})| > q(re^{i\pi/m}) \quad (0 < \theta < \pi/m).
\]
(2.19)

If \( a=0 \), it is obvious that (2.19) is true for \( q(z) = (1+(1-2B)z^m)/(1-z^m) \).

If \( a>0 \), we have
\[
q(z) = (z(1-z^m)^{-2(1-\beta)/m}) \cdot (1-u^{-1})^{-2(1-\beta)/m},
\]
which implies that
\[
|q(re^{i\theta})| > \frac{a(r(1+2r \cos \theta t + r^2 - m)^{-2(1-\beta)/m})}{m}.
\]
Let \( I(\theta) = (1+2r \cos \theta t + r^2 - m)^{-2(1-\beta)/m} \cdot \frac{\alpha}{m} \).

We can verify \( I'(\theta) > 0 \) \( (0 < \theta < \pi/m) \), which implies the desired result. The proof of
(2.17) is now complete.

From (2.4) and (2.17), we get the following distortion result.

COROLLARY 3. If \( f(z) \in J_\alpha \), \( |z|=r < 1 \), then
\[
k'_m(\alpha, \beta, re^{i\pi/m}) < \frac{f'(z)}{k_m(\alpha, \beta, r)}.\]

From (2.13), we can also obtain the sharp order of starlikeness for functions in
\( J_m(\alpha, \beta) \).

THEOREM 5. Let \( f(z) \in J_m(\alpha, \beta) \). Then \( f(z) \in J_m(0, s_m(\alpha, \beta)) \), that is, \( f(z) \) is
starlike of order \( s_m(\alpha, \beta) \), where
\[
s_m(\alpha, \beta) = \min_{0 < \theta < 2\pi/m} \text{Re} e^{i\theta} k_m(\alpha, \beta, e^{i\theta}) / k_m(\alpha, \beta, e^{i\theta}) > \beta.
\]

Miller, Mocanu and Reade [1] proved that \( f(z) \) is a convex function if
\( f(z) \in J(\alpha, 0) \) and \( \alpha > 1 \). By making use of theorem 5, we get the following sharp order
of convexity, which is analogous in sharpness to a result in [1].
COROLLARY 4. If \( f(z) \in J_m(\alpha, \beta) \) and \( \alpha > 1 \), then

\[ f(z) \in J_m(1, \beta/a + (1-1/a)s_m(\alpha, \beta)), \]

that is, \( f(z) \) is convex of order

\( \beta/a + (1-1/a)s_m(\alpha, \beta) \) \((\gg \beta)\).

By using the method we used in [3], we can easily get the following covering theorem from (2.4).

THEOREM 6. Let \( w = f(z) \in J_m(\alpha, \beta) \). Then we have the sharp result

\[ f(D) \ni \{ w : |w| < d_m(\alpha, \beta) \}, \]

where

\[ d_m(\alpha, \beta) = \begin{cases} 2-2(1-\beta)/m & (\alpha = 0) \\ F(1/m, 2(1-\beta)/m, 1+1/m; -1) & (\alpha > 0) \end{cases} \]

and \( F \) is the hypergeometric function.

Finally, we note a coefficient inequality, which can be deduced from (2.1) and a similar result on \( J(\alpha, \beta) \) given in [2].

THEOREM 7. Let \( f(z) = z + a_{m+1}z^{m+1} + a_{2m+1}z^{2m+1} + \ldots \in J_m(\alpha, \beta) \), then we have the sharp inequalities

\[ |a_{2m+1} - \lambda a_{m+1}^2| < \begin{cases} \frac{(1-\beta)^2}{m(1+2\alpha)} \left( \frac{2m+6\alpha+2^2-2(\lambda-1)(1+2\alpha)}{m(1+\alpha)^2} + \frac{\beta}{1-\beta} \right) & -\infty < \lambda < a; \\ \frac{(1-\beta)(1+2\alpha)}{m(1+2\alpha)} & a < \lambda < b; \\ \frac{(1-\beta)^2}{m(1+2\alpha)} \left( \frac{(|m-2\lambda)(1+2\alpha)/(m(1+\alpha)^3)-\beta/(1-\beta)}{b < \lambda < \infty} \right) \end{cases} \]

where

\[ a = \frac{1}{2} + \frac{1}{2} \frac{2\alpha}{(1+2\alpha)}, \quad b = \frac{1}{2} + \frac{1}{2} \frac{2\alpha}{(1+2\alpha)} + \frac{1}{2} \frac{1+\alpha}{(1+2\alpha)^2}/((1+2\alpha)(1-\beta)). \]

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