A NOTE ON COMPLEX L₁-PREDUAL SPACES

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ABSTRACT. Some characterizations of complex L₁-predual spaces are proved.

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1. INTRODUCTION.

The aim of this note is to give some characterizations of complex L₁-predual spaces. These are mostly complex analogous of the results proved by Lau [1].

Existing results that we need are given in §2 and the main results in §3.

Throughout the paper, we shall take V to be a complex Banach space, K its dual unit ball which being convex and compact in the w*-topology has a non-empty set of extreme points e K. For real valued bounded function f on K, f stands for its upper envelope. We shall write Γ = {z ∈ ℂ: |z| = 1}. By A₀(K) we shall mean the set of continuous affine functions f on K which are Γ-homogeneous i.e. f(αx) = αf(x) for all x ∈ K and all α ∈ Γ.

NOTATION. If f is a semi-continuous function on K, then we use the notation Sf to mean

\[ Sf(x) = \frac{1}{2} \left( \frac{1}{2} - \int_{-\pi/2}^{\pi/2} \cos \theta f(xe^{i\theta}) d\theta \right) \]

2. SOME USEFUL RESULTS.

In what follows we need the following results.

THEOREM 2.1. For a complex Banach space V, the following are equivalent:

(i) V is L₁-predual.

(ii) If g is l.s.c. concave function on K, such that

\[ \sum_{k=1}^{n} g(\zeta_k x) > 0 \text{ whenever } \zeta_k \in \Gamma, \,(k = 1,2,3,...,n), \]
\[ \sum_{k=1}^{n} \zeta_k = 0 \], then there is an \( a \in A_o(K) \) such that \( g \geq \text{Re} \ a \) on \( K \).

(iii) If \( h \) is a u.s.c. convex function on \( K \), such that even \( Sh(x) < 0 \) for \( x \in K \), then there is an \( a \in A_o(K) \) such that \( h \leq \text{Re} \ a \) on \( K \).

(iv) For u.s.c. convex function \( g \) on \( K \),

\[ \hat{g}(0) < \sup \left\{ \sum_{k=1}^{n} \alpha_k g(\zeta_k x) : x \in K, n \in \mathbb{N}, \alpha_k > 0, \right\} \]

\[ \sum_{k=1}^{n} \alpha_k = 1, \zeta_k \in \Gamma, \sum_{k=1}^{n} \alpha_k \zeta_k = 0. \]

The equivalence of (i) and (ii) is due to Olsen [2] while that of (i), (iii), (iv) is due to Das [3] and Roy [4]. The inequality in (iv) is in fact an equality since the reverse inequality follows from the fact the \( g \leq \hat{g} \) and that \( \hat{g} \) is concave.

The following result is due to Olsen [5].

**THEOREM 2.2.** For a complex Banach space \( V \), the following are equivalent:

(i) \( V \) is \( L_1 \)-predual with \( \partial K \cup \{0\} \) closed.

(ii) If \( f \) is a continuous \( \Gamma \)-homogeneous function on \( K \), then there is a \( v \in V \) such that \( f|_{\partial K} = v|_{\partial K} \).

3. **MAIN RESULTS.**

This section contains the main results.

**THEOREM 3.1.** A complex Banach space \( V \) is \( L_1 \)-predual iff

\[ \hat{f}(0) = \frac{1}{2} \sup \{ Sf(x) + Sf(-x) : x \in K \} \]

for all u.s.c. convex functions \( f \) on \( K \).

**PROOF.** "If" part.

Let us suppose that for u.s.c. convex functions \( f \) on \( K \),

\[ \hat{f}(0) = \frac{1}{2} \sup \{ Sf(x) + Sf(-x) : x \in K \}. \]

We put

\[ \alpha = \sup \left\{ \sum_{k=1}^{n} \alpha_k f(\zeta_k x) : x \in K, n \in \mathbb{N}, \alpha_k > 0, \right\} \]

\[ \sum_{k=1}^{n} \alpha_k = 1, \zeta_k \in \Gamma, \sum_{k=1}^{n} \alpha_k \zeta_k = 0. \]

Then clearly \( f(x) + f(-x) < 2\alpha \) for \( x \in K \). By linearity and canonical positivity of \( S \), \( Sf(x) + Sf(-x) < 2\alpha \) for all \( x \in K \). Then by the hypothesis \( \hat{f}(0) < \alpha \), so that by Theorem 2.1 (iv), \( v \) is \( L_1 \)-predual.

"Only if" part.

Let \( V \) be \( L_1 \)-predual. Then by Theorem 2.1 (iv),

\[ \hat{f}(0) = \sup \left\{ \sum_{k=1}^{n} \alpha_k f(\zeta_k x) : x \in K, n \in \mathbb{N}, \alpha > 0, \right\} \]

\[ \sum_{k=1}^{n} \alpha_k = 1, \zeta_k \in \Gamma, \sum_{k=1}^{n} \alpha_k \zeta_k = 0. \]
We put \( b = \frac{1}{2} \sup \{ Sf(x) + Sf(-x) : x \in K \} \). Since \( f \) is u.s.c, convex and \( Sf(x) + Sf(-x) < 2b \) for all \( x \in K \), we apply Theorem 2.1 (iii), to the functions \( f-b \) to get \( a \in A(K) \) such that \( f-b \in L(K) \). But \( Re a \in A(K) \), so that \( f(0) < b \). Now \( \hat{f}(0) \) being real constant and \( S \) being linear and canonically positive

\[
b > \hat{f}(0) > \frac{1}{2} \left\{ f(x) + f(-x) \right\}
\]

which yields \( b > \hat{f}(0) > \frac{1}{2} \left\{ Sf(x) + Sf(-x) \right\} \).

Thus \( b > \hat{f}(0) > \frac{1}{2} \sup \{ Sf(x) + Sf(-x) : x \in K \} = b \); the theorem is thus proved.

**REMARK.** The "if" part is proved by Roy [4] in a method quite different from ours, but he has failed to prove the converse and has kept the question open.

**PROPOSITION 3.2.** Let \( V \) be a complex \( L_1 \)-predual space. If \( X \subset K \cup \{0\} \) is closed such that \( x \in X \) whenever \( x \in X, \ a \in \Gamma \), then every continuous \( f : X \rightarrow \mathbb{C} \) with \( f(\alpha x) = \alpha f(x) \) can be extended to an \( \tilde{f} \in A(K) \).

**PROOF.** As \( X \) is compact, \( \text{Re} f(x) \) attains infimum \( c \) (say) on \( X \). Clearly \( c < 0 \), since \( f(-x) = -f(x) \). We define a real-valued function \( F \) on \( K \) by

\[
F(x) = \begin{cases} 
\text{Re} f(x), & x \in X, \\
0, & x \notin X.
\end{cases}
\]

Then \( F \) is u.s.c. and convex on \( K \). Let us take \( \zeta_k \in \Gamma, k=1,2,...,n \) such that \( \Sigma_k = 0 \).

If \( \zeta_k = \exp(i\theta_k), 0 < \theta_k < 2\pi \), then

\[
\sum_{k=1}^{n} \cos \theta_k = \sum_{k=1}^{n} \sin \theta_k = 0.
\]

When \( x \in K \setminus X \),

\[
\text{IF}(\zeta_k x) < 0 \quad \text{and when} \quad x \in X, \quad \text{IF}(\zeta_k x) = \Sigma \{ \cos \theta_k \text{Re} f(x) - \sin \theta_k \text{Im} f(x) \} = 0. \quad \text{Thus for all} \quad x \in K, \quad \text{IF}(\zeta_k x) < 0. \quad \text{Hence by Theorem 2.1(ii), there is an} \quad \tilde{f} \in A(K) \text{ such that} \quad f < \text{Re} \tilde{f}. \]

Let \( x_0 \in X \); then \( \text{Re} f(x_0) < \text{Re} \tilde{f}(x_0) \) and \( \text{Re} f(-x_0) < \text{Re} \tilde{f}(-x_0) \) which combined together give \( \text{Re} f(x_0) = \text{Re} \tilde{f}(x_0) \). Again \( \text{Re} f(ix_0) < \text{Re} \tilde{f}(ix_0) \) and \( \text{Re} f(-ix_0) < \text{Re} \tilde{f}(-ix_0) \) together give \( \text{Im} f(x_0) = \text{Im} \tilde{f}(x_0) \). Thus \( f(x_0) = \tilde{f}(x_0) \). Hence \( \tilde{f} \) is the required extension.

**THEOREM 3.3.** A Banach space \( V \) is \( L_1 \)-predual with \( \mathbb{C} \) \( K \cup \{0\} \) closed if and only if every continuous function \( f : K \cup \{0\} \rightarrow \mathbb{C} \) with \( f(\alpha x) = \alpha f(x) \), \( \alpha \in \Gamma \), can be extended to an \( \tilde{f} \in A(K) \).

**PROOF.** "Only if" part.

Proof of this part is almost similar to that of Theorem 3.2 and is left out. In fact we can define an \( F \) as

\[
F(x) = \begin{cases} 
\text{Re} f(x), & x \in \mathbb{C} \cup \{0\}, \\
\inf \{ \text{Re} f(y) : y \in \mathbb{C} \cup \{0\}, x \in K \setminus \mathbb{C} \cup \{0\} \}, & x \in K \setminus \mathbb{C} \cup \{0\},
\end{cases}
\]

which is u.s.c. convex and satisfies all the conditions of Theorem 2.1(ii).
"If" part.
Suppose that the extension property holds. To prove that $V$ is $L_1$-predual with $\partial_e K \cup \{0\}$ closed, we shall show that Theorem 2.2(ii) holds.

So let $h$ be a $\Gamma$-homogeneous continuous function on $K$ and let $f = h|_{\partial_e K}$.

Then $f(\alpha x) = \alpha f(x)$ for all $\alpha \in \Gamma$ and for all $x \in \partial_e K$. So there is a $v \in V$ such that $v|_{\partial_e K} = f$, that is, $v|_{\partial_e K} = h|_{\partial_e K}$. This completes the proof.

REMARK. This theorem is comparable with a characterizing result for Bauer simplex that every continuous function on $\partial_e K$ can be extended to a function in $A(K)$.

REFERENCES