ABSTRACT. The main result we obtain is that given \( \pi : N \rightarrow M \) a \( T^a \)-subbundle of the generalized Hopf fibration \( \tilde{\pi} : H^{2n+a} \rightarrow \mathbb{C}P^a \) over a Cauchy-Riemann product \( M \subset \mathbb{C}P^a \), i.e. \( j : N \subset H^{2n+a} \) is a diffeomorphism on fibres and \( \tilde{\pi} \circ j = \pi \), if \( s \) is even and \( N \) is a closed submanifold tangent to the structure vectors of the canonical \( \mathcal{R} \)-structure on \( H^{2n+a} \) then \( N \) is a Cauchy-Riemann submanifold whose Chen class is non-vanishing.

KEY WORDS AND PHRASES. Principal toroidal bundle, \( \mathcal{R} \)-manifold, generalized Hopf fibration, framed C.R. submanifold, characteristic form.

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1.- INTRODUCTION AND STATEMENT OF RESULTS.

As a tentative of unifying the concepts of complex and anti-invariant submanifolds of an almost Hermitian manifold, A. BEJANCU, [1], has introduced the notion of Cauchy-Riemann (C.R.) submanifold. This has soon proved to possess a largely rich number of geometrical properties; e.g. by a result of D.E.BLAIR & B. Y.CHEN, [2], any C.R. submanifold of a Hermitian manifold is a Cauchy-Riemann manifold, in the sense of A.ANDREOTTI & C.D.HILL, [3].

Let \( M^{2n+a} \) be a real \((2n+s)\)-dimensional manifold carrying a metrical \( f \)-structure \( (f, \xi_a, \eta_a, \mathcal{R}) \), \( 1 \leq a \leq s \), with complemented frames, cf. [4]. A submanifold \( j : N \rightarrow M^{2n+a} \) is said to be a framed C.R. submanifold if it is tangent to each structure vector \( \xi_a \) of \( M^{2n+a} \) and it carries a pair of complementary (with respect to \( G = j^* \mathcal{R} \)) smooth distributions \( \mathcal{D}, \mathcal{D}^\perp \) such that \( f_x(\mathcal{D}^\perp_x) \subseteq \mathcal{D}^\perp_x \), \( f_x(\mathcal{D}_x) \subseteq \mathcal{T}_x(N)^\perp \), for all \( x \in N \), where \( \mathcal{T}(N)^\perp \rightarrow N \) stands for the normal bundle of \( j \). Cf. I.MIHAI, [5], L.ORNEA, [6]. Since \( f \)-structures are known to generalize both almost complex \((s=0)\) structures and almost contact \((s=1)\) structures, the notion of framed C.R. submanifold containes those of a C.R. submanifold (see e.g. [7], p.83) of an almost Hermitian manifold and of a
contact C.R. submanifold (see e.g. [7], p.48) of an almost contact metrical manifold.

Let $\tilde{\pi} : H^{2n+s} \to CP^n$ be the generalized Hopf fibration, as given by D.E.BLAIR, [8]. Leaving definitions momentarily aside we may formulate the following:

**THEOREM A**

i) Let $N$ be a framed C.R. submanifold of an $\mathcal{F}$-manifold $M^{2n+s}$. Then the $\xi$-anti-invariant distribution $\mathcal{D}^\perp$ of $N$ is completely integrable.

ii) Any framed C.R. submanifold of $H^{2n+s}$, (carrying the standard $\mathcal{F}$-structure) is either a C.R. submanifold ($s$ even) or a contact C.R. submanifold ($s$ odd). The converse holds.

iii) Let $N$ be an $\xi$-invariant (i.e. $\mathcal{D}^\perp = 0$) submanifold of $H^{2n+s}$. Then $N$ is totally-geodesic if and only if it is an $\mathcal{F}$-manifold of constant $\xi$-sectional curvature $1 - \frac{3}{4}s$.

iv) Any $\xi$-invariant submanifold of $H^{2n+s}$ having a parallel second fundamental form is totally-geodesic.

It is known that compact regular contact manifolds are $S^1$-principal fibre bundles over symplectic manifolds, cf. W.M. BOOTHBY & H.C.WANG, [9]. Eversince this (today classical) paper has been published, several "Boothby-Wang type" theorems have been established, cf. e.g. A.MORIMOTO, [10], for the case of normal almost contact manifolds, S.TANNO, [11], for contact manifolds in the non-compact case; more recently, we may cite a result of I.VAISMAN, [12], asserting that compact generalized Hopf manifolds with a regular Lee field may be fibred over Sasakian manifolds, etc.

There exists today a large literature, cf. K.YANO & M.KON, [7], concerned with the study of the geometry (of the second fundamental form) of a C.R. submanifold of a Kaehlerian ambient space. In particular, following the method of Riemannian fibre bundles (such as introduced by H.B.LAWSON, [13], towards studying submanifolds of complex space-forms, and developed successively by Y.MAEDA, [14], M.OKUMURA, [15]), K.YANO & M.KON, [16], have taken under study contact C.R. submanifolds of a Sasakian manifold $M^{2n+1}$ (where $M^{2n+1}$ is previously fibred over a Kaehlerian manifold $M^n$ which are themselves $S^1$-fibrations over C.R. submanifolds of $M^n$.

The last piece of the mosaic we are going to mend is the concept of canonical cohomology class (here after referred to as the Chen class) of a C.R. submanifold. Cf. B.Y.CHEN, [17], with any C.R. submanifold $M$ of a Kaehlerian manifold there may be associated a cohomology class $c(M) \in H^p(M; R)$, where $p$ stands for the complex dimension of the holomorphic distribution of $M$. Although the canonical Hermitian structure (cf. [18]) of $H^{2n+s}$ is never Kaehlerian (cf. [8], p.174) we show that the Chen class of a C.R. submanifold may be constructed as well and obtain the following:

**THEOREM B**

Let $j : N \to H^{2n+s}$ be a closed (i.e. compact, orientable) submanifold tangent to the vector fields $\xi_a$, $1 \leq a \leq s$, of the canonical $\mathcal{F}$-structure on $H^{2n+s}$ and assume there exists a $T^s$-principal bundle $\pi : N \to M$ over a Cauchy-
Riemann product \((M, \mathbb{D}, \mathbb{D}^\perp)\), \(i : M \to \mathbb{C}P^n\), \((\mathbb{D})\) is the holomorphic distribution, such that \(\pi \circ j = 0\) and \(j\) is a diffeomorphism on fibres. If \(s\) is even then \(N\) is a C.R. submanifold whose totally-real foliation is normal to the characteristic field of \(H^{2n+s}\) and whose Chen class \(c(N) \in H^{2n+s}(N; \mathbb{R})\), \(p = \dim_c \mathbb{D}\), is non-vanishing.

2.- NOTATIONS, CONVENTIONS AND BASIC FORMULÆ.

Let \(M^{2n+s}\) be a real \((2n+s)\)-dimensional \(C^\infty\)-differentiable connected manifold. Let \(f\) be an \(f\)-structure on \(M^{2n+s}\), i.e. a \((1,1)\)-tensor field such that \(f^0 + f = 0\) and rank\((f) = 2n\) everywhere on \(M^{2n+s}\), cf. K.YANO, [19]. Assume that \(f\) has complemented frames, i.e. there exist the differential 1-forms \(\eta^i\) and the dual vector fields \(\xi^i\) on \(M^{2n+s}\), i.e. \(\eta^i(\xi^j) = \delta_{ij}, 1 \leq a,b \leq s\), such that the following formulae hold:

\[
\eta^i \circ f = 0, \quad f(\xi^i) = 0, \quad \xi^2 = -I + \eta^a \otimes \xi^a. \quad (2.1)
\]

Throughout, one adopts the convention \(\eta^i = \eta^{a*}\), \(\xi^i = \xi^{a*}\). The \(f\)-structure is normal if \([f, f] + (d\eta^i) \otimes \xi^i = 0\), where \([f, f]\) denotes the Nijenhuis torsion of \(f\), see e.g. H.NAKAGAWA, [20]. Let \(\mathbb{B}\) be a compatible Riemannian metric on \(M^{2n+s}\), i.e. one satisfying:

\[
\mathbb{B}(fX, fY) = \mathbb{B}(X, Y) - \eta^a(X) \eta^{a*}(Y). \quad (2.2)
\]

Compatible metrics always exist, cf. D.E.BLAIR, [4]. Such \((f, \xi^i, \eta^i, \mathbb{B})\) has often been called a metrical \(f\)-structure with complemented frames. Let \(E(X, Y) = \mathbb{B}(X, Y)\) be its fundamental 2-form. Throughout we assume \(M^{2n+s}\) to be an \(\mathbb{R}\)-manifold, cf. the terminology in [4], i.e. the given \(f\)-structure is normal, its fundamental 2-form is closed and there exist \(s\) smooth real-valued functions \(\alpha^s \in C^\infty(M^{2n+s}), 1 \leq a \leq s\), such that:

\[
d \eta^i = \alpha^a \xi^a. \quad (2.3)
\]

We shall need, cf. [4], [21], the following result. Let \(M^{2n+s}\), \(n > 1\), be a connected manifold carrying the \(\mathbb{R}\)-structure \((f, \xi^i, \eta^i, \mathbb{B})\), \(1 \leq a \leq s\). Then \(\alpha^s\) are real constants, \(\xi^i\) are Killing vector fields (with respect to \(\mathbb{B}\)) and the following relations hold:

\[
(D_X \xi^i) = -\frac{1}{2} \alpha^a f X \quad (2.4)
\]

\[
(D_X f) Y = \frac{1}{2} \alpha^a \{[\mathbb{B}(X, Y) - \eta^a(X) \eta^{a*}(Y)] \xi^b - [X - \eta^a(X) \xi^{a*}] \eta^b(Y)\} \quad (2.5)
\]

for any tangent vector fields \(X, Y\) on \(M^{2n+s}\). Here \(D\) denotes the Riemannian connection of \((M^{2n+s}, \mathbb{B})\), and \(\alpha^s = \alpha^a, 1 \leq a \leq s\).

Let \(M^{2n+s}\) be an \(\mathbb{R}\)-manifold with the structure tensors \((f, \xi^i, \eta^i, \mathbb{B})\). Let \(\mathfrak{m}\) be the smooth s-distribution on \(M^{2n+s}\) spanned by \(\xi^i\), \(1 \leq a \leq s\). By normality one has \([\xi^a, \xi^b] = 0\), i.e. \(\mathfrak{m}\) is involutive. If both \(\mathfrak{m}\) and the structure vector fields \(\xi^i\) are regular (in the sense of R.PALAIAS, [22]) then the \(\mathbb{R}\)-structure itself is termed regular. We shall need the main result of D.E. BLAIR & G.D.LUDDEN & K.YANO, ([21], p.175). That is, let \(M^{2n+s}\) be a compact connected \((2n+s)\)-dimensional, \(n > 1\), \(\mathbb{R}\)-manifold; then there is a \(T^s\)-principal fibre bundle \(\pi : M^{2n+s} \to M^{2n} = M^{2n+s}/\mathfrak{m}\) and \(M^{2n}\) is a Kaehlerian.
manifold. Here $M^{2n}$ denotes the leaf space of the $s$-dimensional foliation $\mathcal{F}$ and $T^s$ is the $s$-torus. Also, cf. ([21], p.178), $\gamma = (\eta^s_1, \ldots, \eta^s_s)$ is a connection $s$-form in $M^{2n+a}(M^{2n}, \tilde{\pi}, T^s)$. If $X$ is a tangent vector field on $M^{2n}$, let $X^H$ denote its horizontal lift with respect to $\gamma$. The Kaehlerian structure $(J, g)$ of $M^{2n}$ is expressed by:

$$J X = \tilde{\pi} \ast f X^H \quad (2.6)$$

$$\tilde{g}(X, Y) = \Theta(X^H, Y^H). \quad (2.7)$$

Let $\mathcal{D}$ be the smooth $2n$-distribution on $M^{2n+s}$ defined by the Pfaffian equations $\eta^s_a = 0, 1 \leq a \leq s$. Then $\mathcal{D}$ is precisely the horizontal distribution of $\gamma$. Since $\eta^s_a \circ f = 0$, the $f$-structure preserves the horizontal distribution.

Therefore (2.6) may be also written $(J X)^H = f X^H$. Let $\nabla$ be the Riemannian connection of $(M^{2n}, \tilde{\pi})$. By ([21], p.179) one has:

$$D_{X^H} Y^H = (\nabla X Y)^H + \frac{1}{2} \alpha^s \Theta(f X^H, Y^H) \xi^s \quad (2.8)$$

**REMARK**

Let $\pi : N \to M$ be a Riemannian submersion, cf. B. O'NEILL, [23]. Then $\text{Ker}(\pi_\ast)$ is the *vertical distribution*, while its complement (with respect to the Riemannian metric of $N$) is the *horizontal distribution* of the Riemannian submersion. As to our $\tilde{\pi}: M^{2n+s} \to M^{2n}$ a number of important coincidences occur.

Firstly, if $M^{2n}$ is assigned the Riemannian metric (2.7), then $M^{2n+s} \to M^{2n}$ is a Riemannian submersion. Moreover $\mathcal{F} = \text{Ker}(\tilde{\pi}_\ast)$ and therefore the horizontal distribution of the Riemannian submersion is precisely $\mathcal{D}$.

Let $N$ be an $(m+s)$-dimensional submanifold of $M^{2n+s}$, and $M$ an $m$-dimensional submanifold of $M^{2n}$, such that there exists a fibering $\pi : N \to M$ such that $\tilde{\pi} \circ j = i \circ \pi$ and $j$ is a diffeomorphism on fibres. Both $i : M \to M^{2n}$, $j : N \to M^{2n+s}$ stand for canonical inclusions. Let $g = i^* \tilde{\pi}^* g$, $G = j^* \Theta$ be the induced metrics on $M$ and $N$, respectively. Also we denote by $\nabla$, $D$ the corresponding Riemannian connections of $(M, g)$ and $(N, G)$, respectively. Let $B$ (resp. $h$) be the second fundamental form of $i$ (resp. $j$) and denote by $A$ (resp. $W$) the Weingarten forms. Let $T(M) \perp \to M$ (resp. $T(N) \perp \to N$) be the normal bundle of $i$ (resp. $j$). We put $\xi^s_x = \text{tan} (\xi^s_x)$, $\xi^s_x \perp = \text{nor} (\xi^s_x)$, where $\text{tan}_x$, $\text{nor}_x$ stand for the projections associated with the direct sum decomposition $T_x(M^{2n+s}) = T_x(N) \oplus T_x(N) \perp$, $x \in N$. Then the Gauss and Weingarten formulae, (cf. e.g. [24], p.39-40), of $i$, $j$ and our (2.8) lead to:

$$D_{X^H} Y^H = (\nabla X Y)^H + \frac{1}{2} \alpha^s \Theta(f X^H, Y^H) \xi^s \quad (2.9)$$

$$h(X^H, Y^H) = B(X, Y)^H + \frac{1}{2} \alpha^s \Theta(f X^H, Y^H) \xi^s \quad (2.10)$$

$$W_{Y^H} Y^H = (A_{Y^H} X)^H - \frac{1}{2} \alpha^s \Theta(f X^H, V^H) \xi^s \quad (2.11)$$

$$D^{\perp}_{X^H} V^H = (\nabla^{\perp} X V)^H + \frac{1}{2} \alpha^s \Theta(f X^H, V^H) \xi^s \quad (2.12)$$

for any tangent vector fields $X$, $Y$ on $M$, respectively any cross-section $V$ in $T(M) \perp \to M$. Here $\nabla$, $D$ stand for the normal connections of $i$, $j$. Of course, towards obtaining our (2.9) - (2.12) one exploits the fact that $(i_\ast X)^H$ is tangent to $N$, while $V^H$ is a cross-section in $T(N) \perp \to N$. 


REMARKS
1) Let $H(i) = \frac{1}{m} \text{Trace}(B)$ (resp. $H(j) = \frac{1}{m+s} \text{Trace}(h)$) be the mean curvature vector of $i$ (resp. $j$). As an application of our (2.9) - (2.12) one may derive:

$$(m+s) H(j) = m H(i)^N + \sum_{s=1}^{s} \left[ \frac{1}{2} \alpha^s \text{nor}(\xi^s) - D_{\xi^s} \right]$$

provided that $\{\xi^s; 1 \leq s \leq s\}$ consists of mutually orthogonal unit vector fields. In particular, if $N$ is tangent to each structure vector $\xi^s$, then $N$ is minimal if and only if $M$ is minimal. Indeed, if $X$ is tangent to $N$, then (2.4) and the Gauss - Weingarten formulae lead to:

$$D_{X^s} \xi^s = W_X \bot X - \frac{1}{2} \alpha^s \tan(f X)$$

$$h(X, \xi^s) + D_{\xi^s} \xi^s = - \frac{1}{2} \alpha^s \text{nor}(f X).$$

Now, if $\{\xi^s; 1 \leq s \leq s\}$ are orthonormal, one uses a frame $\{X_i, \xi^s\}$ (where $\{X_i; 1 \leq i \leq m\}$ is an orthonormal tangential frame of $M$) such as to compute $H(j)$.

2) Generally, if $N$ is a submanifold of the manifold $M^{2n+s}$ and $N$ is normal to some $\xi^s$ with $\alpha^s = 0$ then tangent spaces at points of $N$ are $\bot$-anti-invariant, i.e. $f_x(T_x(N)) \subseteq T_x(N) \bot x \in N$. Indeed, by (2.4) and the Weingarten formula of $N$ in $M^{2n+s}$, one has $\mathcal{B}(\alpha^s, f X, Y) = -2 \mathcal{B}(D_{X^s} \xi^s, Y) = 2 \mathcal{B}(W_x \bot X, Y)$ where from $W_x \bot x = 0$ and $f X$ is normal to $N$.

3. ORIFOLDS AS HERMITIAN OR NORMAL ALMOST CONTACT METRICAL MANIFOLDS.

We denote by $\mathbb{P}^n$ the complex projective space with constant holomorphic sectional curvature 1 (with Fubini - Study metric) and complex dimension $n$, and by $S^{2n+1}$ the (2n+1)-dimensional unit sphere carrying the standard Sasakian structure. Let $\pi^1: S^{2n+1} \rightarrow \mathbb{P}^n$ be the Hopf fibration and set $H^{2n+s} = \{(p_1, ..., p_s) \in S^{2n+1} \times \cdots \times S^{2n+1} \mid \pi^1(p_1) = \cdots = \pi^1(p_s)\}$.

We define a principal toroidal bundle by the commutative diagram:

$$\begin{array}{ccc}
H^{2n+s} & \xrightarrow{\hat{\Lambda}} & S^{2n+1} \times \cdots \times S^{2n+1} \\
\pi \downarrow & & \downarrow \pi^1 \times \cdots \times \pi^1 \\
cP^n & \xrightarrow{\Lambda} & cP^n \times \cdots \times cP^n
\end{array}$$

where $\Lambda$ denotes the diagonal map, while $\hat{\Lambda}$ stands for the canonical inclusion. Let $\eta'$ be the standard contact 1-form on $S^{2n+1}$. We put $\eta^s = \hat{\Lambda}^* \Lambda^* \eta'$, $1 \leq a \leq s$ where $\Lambda^s: S^{2n+1} \times \cdots \times S^{2n+1} \rightarrow S^{2n+1}$ are natural projections. Let $\Omega$ be the Kaehler 2-form of $cP^n$. Then on one hand $\gamma = (\eta^1, ..., \eta^s)$ is a connection 1-form in $H^{2n+s}(cP^n, \hat{\pi}, \pi^1)$, and on the other $d\eta' = \hat{\pi}^* \Omega$, such that one may apply theorem 3.1 of [8], (p.163) such as to yield a natural $\mathcal{R}$-structure on $H^{2n+s}$. (Cf also [4], p.173). Let $(f, \xi^s, \eta^s, \mathcal{B})$ be the canonical $\mathcal{R}$-structure.
of $H^{2n+s}$. If $s$ is even one sets:

$$\mathcal{J} = \mathcal{J} + \sum_{i=1}^{s} \{ \eta_i \otimes \xi_i^* - \eta_i^* \otimes \xi_i \}$$

(3.1)

where $i = i + \frac{s}{2}$, $1 \leq i \leq \frac{s}{2}$. If $s$ is odd, one labels the 1-forms $\eta_i$ as follows:

$$\eta_0, \eta_1, \eta_{s-1}, \eta_s, \eta_{s+1} = i + r, 1 \leq i \leq r, s = 2r+1,$$

and similarly for the tangent vector fields $\xi_i$. We consider:

$$\phi = \mathcal{J} + \sum_{i=1}^{r} \eta_i \otimes \xi_i^* - \eta_i^* \otimes \xi_i.$$  

(3.2)

The characteristic 1-form of $H^{2n+s}$, $s$ even, is defined by:

$$\omega = 2 \sum_{i=1}^{s/2} \{ \eta_i - \eta_i^* \}.$$  

(3.3)

Let $B = \omega^+$ be the characteristic field, where $+$ means raising of indices by $\theta$.

REMARKS

1) If $s$ is even then $(H^{2n+s}, \mathcal{J}, \theta)$ is a Hermitian non-Kaehlerian manifold and its characteristic form is parallel. Indeed, if $s$ is even, then $\mathcal{J}$ given by (3.1) is a complex structure and $(H^{2n+s}, \mathcal{J}, \theta)$ turns to be a Hermitian manifold, (cf. prop.4.1 in [8], p.174). Let $F(X, Y) = \theta(X, \mathcal{J} Y)$ be its Kaehler 2-form. By (3.1) it follows that $\tilde{F} = F - 2 \sum_{i=1}^{s/2} \eta_i \wedge \eta_i^*$; consequently (3.3)

$$dF = \omega \wedge \tilde{F}$$

(3.4)

leads to

$$i.e. \theta \text{ is not a Kaehler metric. Now our (2.4) yields } \theta \omega = \frac{1}{2} \sum_{i=1}^{s/2} (\alpha_i - \tilde{\alpha}_i);$$

on an arbitrary $\mathcal{F}$-manifold, provided $s$ is even. Yet for $H^{2n+s}$ one has $\alpha_i - \ldots - \alpha_s$ (cf.[8],p.173), i.e. $\omega$ is parallel.

2) Since $d \eta^* = \bar{\pi}^{*} \Omega$, $1 \leq a \leq s$, it follows that $\omega$ is closed. Therefore $H^{2n+s}$, $s$ even, admits the canonical foliation $\mathcal{F}$ defined by the Pfaffian equation $\omega = 0$. Each leaf of $\mathcal{F}$ is a totally-geodesic real hypersurface normal to the characteristic field of $H^{2n+s}$.

3) Consider the submanifolds $i : M \to \mathbb{CP}^n$ and $j : N \to H^{2n+s}$ and assume that a $T^s$-subbundle $\pi : N \to M$ of the generalized Hopf fibration, i.e. $\pi \circ j = i \circ \pi$ and $j$ is a diffeomorphism on fibres. Suppose $N$ is tangent to the structure vectors $\xi_s$ of the $\mathcal{P}$-manifold $H^{2n+s}$. Then $M$ is a C.R. submanifold of $\mathbb{CP}^n$ if and only if $N$ is either a C.R. submanifold of $(H^{2n+s}, \mathcal{J}, \theta)$ or a contact C.R. submanifold of $(H^{2n+s}, \phi, \xi_0, \eta_0, \theta)$. Note firstly that, if $s$ is odd, then $(\phi, \xi_0, \eta_0, \theta)$ is a normal almost contact metrical (a. ct. m.) structure on $H^{2n+s}$, (cf. [8], p.175). If $\xi_s = 0$, $1 \leq a \leq s$, and $s$ is even then:

$$\mathcal{J} \xi_i = \xi_i^*, \quad \mathcal{J} \xi_s = - \xi_s, \quad \mathcal{J} X^H = (J X)^H$$

(3.5)

for any tangent vector field $X$ on $M$, cf.(2.6). Let us define $\mathcal{P} Y = \tan(\mathcal{J} Y)$, $\mathcal{P} Y = \nor(\mathcal{J} Y)$, for any tangent vector field $Y$ on $N$. Then:

$$\mathcal{P} \xi_i = 0, \quad \mathcal{P} \xi_s = 0, \quad \mathcal{P} X^H = (F P X)^H$$

(3.6)

where $F, P$ are defined by (1.1) in [7] (p.76). Suppose for instance that $(M, \theta, \mathcal{D}^\perp)$ is a C.R. submanifold of $\mathbb{CP}^n$. Then $P$ is $\theta$-valued, while $F$ vanishes on
By (3.6) one has $\mathcal{D}^\perp = \mathcal{D} = 0$, and thus one may apply theor. 3.1 in [7] (p.87), such as to conclude that $N$ is a C.R. submanifold of $(\mathbb{H}^{2n+4}, \mathcal{J}, \mathcal{B})$. Note that, although stated for submanifolds in Kaehlerian manifolds, theor. 3.1 of [7] (p.87) actually holds for the general case of an arbitrary almost Hermitian ambient space. The case $s$ odd follows similarly from theor. 2.1 of [7] (p.55) which may be easily refined from the Sasakian case to the general case of a. ct. m. structures.

4) Let $(M, \mathcal{D}, \mathcal{D}^\perp)$ be a C.R. submanifold of $\mathbb{C}P^n$, where $\mathcal{D}$ (resp. $\mathcal{D}^\perp$) denotes the holomorphic (resp. totally-real) distribution. Let $\pi : N \to M$ be a $T^1$-bundle as in Remark 3). Let $\mathcal{D}_N$, $\mathcal{D}^\perp_N$ be the holomorphic and totally-real (resp. the $\varphi$-invariant and $\varphi$-anti-invariant) distributions of $N$, provided that $s$ is even (resp. $s$ is odd). Let $\mathcal{U}_{N,x}$ the natural projection on the first term of the direct sum decomposition $T_x(N) = \mathcal{D}_{N,x} \oplus \mathcal{D}_{N,x}^\perp$, $x \in N$. Cf. (3.7) in [7] (p.86), (resp. cf. (2.10) in (7) (p.53)) if $s$ is even (resp. if $s$ is odd) then $\mathcal{U}_N$ is expressed by $\mathcal{U}_N = - \mathcal{D}^2$ (resp. by $\mathcal{U}_N = - \mathcal{D}^2 + \eta_0 \otimes \xi_0$) where $\mathcal{D}Y = \tan(\mathcal{J}Y)$, (resp. $\mathcal{D}Y = \tan(\varphi Y)$). In both cases one has:

$$\mathcal{U}_N \xi_a = \xi_s, \quad 1 \leq a \leq s, \quad \xi_a X^\mathcal{H} = (\mathcal{J}X)^\mathcal{H}$$

(3.7)

where $\mathcal{J} = - P^2$. As the sum $\mathcal{D}^\perp + \mathfrak{M}_x$, $x \in N$, is direct one obtains $\mathcal{D}^\perp_{N,x} = \mathcal{D}^\perp_x \oplus \mathfrak{M}_x$, $x \in N$. Indeed, one inclusion follows from (3.7). Conversely, let $X' \in \mathcal{D}^\perp_N$, then $X' = (\mathcal{J}X)^\mathcal{H} + (\mathcal{J}X)^\mathcal{H} \lambda^s \xi_s$, $\lambda^s \in C^\infty(N)$, $\mathcal{J} - 1 = - \mathcal{J}$. By applying $\mathcal{J}N$ to both members one proves $X' \in \mathcal{D}^\perp \oplus \mathfrak{M}$. It is also straightforward that $(\mathcal{D}^\perp)^\mathcal{H} = \mathcal{D}^\perp_N$.

4.- FRAMED CAUCHY-RIEMANN SUBMANIFOLDS

S. GOLDBERG, [25], has inaugurated a program of unifying the treatment of the cases $s$ even, and $s$ odd, and studied $\mathcal{F}$-invariant submanifolds of codimension 2 of an $\mathfrak{F}$-manifold. To make the terminology precise, let $(N, \mathcal{D}, \mathcal{D}^\perp)$ be a framed C.R. submanifold of $M^{2n+4}$, we call $N$ an $\mathcal{F}$-invariant (resp. $\mathcal{F}$-anti-invariant) submanifold if $\mathcal{D}$ (resp. if $\mathcal{D}^\perp$), for any $x \in N$.

Let $M^{2n+4}$ be an $\mathfrak{F}$-manifold; let $x \in M^{2n+4}$ and $\mathfrak{p} \subseteq T(M^{2n+4})$ a 2-plane. (Cf. [8], p.159), $\mathfrak{p}$ is an $\mathcal{F}$-section if it is spanned by $\{X, \mathcal{J}X\}$ for some unit tangent vector $X \in \mathcal{L}_x$. The Riemannian sectional curvature of $(M^{2n+4}, \mathcal{D})$ restricted to $\mathcal{F}$-sections is referred to as the $\mathcal{F}$-sectional curvature of the $\mathfrak{F}$-manifold. (Cf. also [21], p.183).

At this point we may establish i) of theor. A. Let $X, V$ be respectively a tangent vector field on $N$ and a cross-section in $T(N)^\perp \to N$. We set $P = \tan(fX)$, $F = \text{nor}(fV)$ and $f V = \text{nor}(fV)$. The following identities hold as direct consequences of definitions:

$$P^2 + t F = - I + \eta_0 \otimes \xi_0, \quad F P + f F = 0, \quad P t + t f = 0,$$

$$F t + f^2 = - I, \quad f \mathcal{J} = P \mathcal{J}, \quad F \mathcal{J} = 0, \quad \mathcal{J} \mathcal{J} = 0.$$ (4.1)

Using (2.5) and the Gauss - Weingarten formulae of $N$ in $M^{2n+4}$ one obtains:

$$(D_X P) Y = W_{FY} X + t h(X, Y) +$$

$$+ \frac{1}{2} \xi_s \left\{ [G(X, Y) - \eta_0(X) \eta^s(Y)] \xi_s - [X - \eta_0(X) \xi_s] \eta_s(Y) \right\}$$

(4.2)

for any tangent vector fields $X, Y$ on $N$. Let $X, Y \in \mathcal{D}^\perp$. As $D$ is torsion-free
and by (4.2) one obtains:

\[ P[X, Y] = WFX Y - WFY X + \alpha^\bullet \left\{ \frac{1}{2} (X \wedge Y) \xi^\bullet + (\eta^\bullet \wedge \eta^\bullet) (X, Y) \xi^\bullet \right\} \]  

(4.3)

At this point we may establish the following:

**LEMMA**

Let \((N, \mathcal{D}, \mathcal{D}^\perp)\) be a framed C.R. submanifold of the \(\mathcal{R}\)-manifold \(M^{2n+s}\). Then:

\[ WFX Y = WFX X + \frac{1}{2} \alpha^\bullet \left\{ \eta^\bullet(X) Y - \eta^\bullet(Y) X - [\eta^\bullet(X) \eta^\bullet(Y) - \eta^\bullet(Y) \eta^\bullet(X)] \xi^\bullet \right\} \]  

(4.4)

for any \(X, Y \in \mathcal{D}^\perp\).

**Proof.** By (4.1), \(P\) vanishes on \(\mathcal{D}^\perp\). Using (4.2) for any \(X, Y \in \mathcal{D}^\perp, Z \in T(N)\), one has:

\[ 0 = G((D Z P)X, Y) = G(WFX Z, Y) + G(t h(Z, X), Y) + \]

\[ + \frac{1}{2} \alpha^\bullet \left\{ G(Z, X) \eta^\bullet(Y) - G(Z, Y) \eta^\bullet(X) + [\eta^\bullet(X) \eta^\bullet(Y) - \eta^\bullet(Y) \eta^\bullet(X)] \eta^\bullet(Z) \right\} \]

and finally \(G(t h(Z, X), Y) = - G(WFY X, Z)\) leads to (4.4).

By (4.3) and the above lemma we conclude \(P[X, Y] = 0\), i.e. \(\mathcal{D}^\perp\) is involutive.

Let us prove now ii) in theor. A. We analyse for instance the case \(s\) even. Let \(N\) a framed C.R. submanifold of \(H^{2n+s}\). Let

\[ \mathcal{P} = P + \sum_{i=1}^{s/2} \eta_i \otimes \xi_i, \quad \mathcal{P}^\perp = F \]  

(4.5)

Next \(\mathcal{P}^\perp = F P = 0\), and one applies theor.3.1 of [7], p.87. The case \(s\) odd being similar is left as an exercise to the reader. To prove the converse of ii) in theor. A we need to characterize framed C.R. submanifolds as follows.

Let \(N\) be a framed C.R. submanifold of an \(\mathcal{R}\)-manifold \(M^{2n+s}\). Then (4.1) leads to \(P \perp = F, F P = 0, F F = 0\), etc. One obtains the following statement. Let \(N\) be a submanifold of the \(\mathcal{R}\)-manifold \(M^{2n+s}\) such that \(N\) is tangent to the structure vectors \(\xi_i\). Then \(N\) is a framed C.R. submanifold of \(M^{2n+s}\) if and only if \(F P = 0\). We have proved the necessity already. Viceversa, let us put by definition \(\perp = - P^2 + \eta^s \otimes \xi^s, \perp = 1 - \perp\). Since \(F P = 0\), the projections \(\perp, \perp\) make \(N\) into a framed C.R. submanifold, Q.E.D. Now the converse of ii) in theor. A is easily seen to hold, i.e. both C.R. submanifolds of \((H^{2n+s}, \mathcal{F}, \mathcal{D})\), \(s\) even, and contact C.R. submanifolds of \((H^{2n+s}, \varphi, \xi^\circ, \eta^\circ, \mathcal{D})\), \(s\) odd, are framed C.R. submanifolds.

**REMARKS**

1) Let \((N, \mathcal{D}, \mathcal{D}^\perp)\) be a framed C.R. submanifold of \(H^{2n+s}\). By (4.5) one obtains:

\[ \mathcal{P}^2 = P^2 - \eta^s \otimes \xi^s. \]  

(4.6)

Now the notion of framed C.R. submanifold appears to be essentially on old concept. For not only \(N\) becomes a C.R. submanifold of the Hermitian manifold \(H^{2n+s}\), if for instance \(s\) is even, but its holomorphic and totally-real distributions are precisely \(\mathcal{D}, \mathcal{D}^\perp\). Indeed, by (4.6) one has \(\perp = \perp\), Q.E.D.

2) Due to (3.4) there is a certain similarity between \(\mathcal{R}\)-manifolds and locally conformal Kaehler manifolds, cf. P.LIBERMANN, [26]. See also [12]. For instance, we may use the ideas in [2] (cf. also theor. 3.4 of [7], p.89) to
give an other proof of the integrability of the f-anti-invariant distribution of a framed C.R. submanifold. Indeed, let N be a framed C.R. submanifold of $H^{2n+\ast}$, s even. Let $X \in \mathfrak{D}$, $Z, W \in \mathfrak{D}^\perp$. By (3.4) one has $0 = (G \tilde{F})(X, Y, W) = G([Z, W], J X)$. Hence $[Z, W] \in \mathfrak{D}^\perp$. Note that, although N is C.R. in the usual sense one could not apply theor.3.4 or theor.4.1 of [7] (p.89-90) since $H^{2n+\ast}$ is neither locally conformal Kaehler nor Kaehler.

To establish iii) let N be an f-invariant submanifold of $H^{2n+\ast}$. As a consequence of (2.5), for any tangent vector fields $X, Y$ on N one has:

$$\langle D_X fY, \xi \rangle = \frac{1}{2} \alpha^a \{ [G(X, Y) - \eta_a(X) \eta_b(Y)] \xi_a - [X - \eta_a(X) \xi_a] \eta_b(Y) \} \tag{4.7}$$

$$h(X, fY) = f h(X, Y). \tag{4.8}$$

Let $k(X, Y)$ be the Riemannian sectional curvature of the 2-plane spanned by the orthonormal pair $\{X, Y\}$ on N; using the Gauss equation, i.e. equation (2.6) in [24], (p.45), and the notations in [4], (p.161), i.e. $H(X) = k(X, fX), X \in \mathfrak{D},$ one obtains:

$$1 - \frac{\alpha}{4} = H(X) + 2 || h(X, X) ||^2 \tag{4.9}$$

as $H^{2n+\ast}$ has constant f-sectional curvature, (cf.[8], p.173). By (2.15) and f-invariance one has $h(X, \xi_a) = -\frac{1}{2} \alpha \text{nor}(fX) = 0$; a standard argument based on (4.8) leads to the proof.

To prove iv) one uses $D h = 0$, (2.15) and f-invariance, i.e. one has $h((D_X \xi_a, Y) = 0$. Thus $\alpha \text{ h}(fX, Y) = 0$, by (2.14). For some $\alpha = 0$ one uses (4.7). Finally, apply once more f and notice that $\eta_a$ vanish on normal vectors. Thus $h = 0$.

REMARK

Let $\mathcal{F}$ be the canonical foliation of $H^{2n+\ast}$. Let N be a framed C.R. submanifold of $H^{2n+\ast}$, as above. Then $\mathfrak{D}^\perp \subseteq \mathcal{F}$, i.e. the totally-real foliation of N (regarded as a C.R. submanifold, s even) is normal to the characteristic field $\sum_{i=1}^{2n} (\xi_i, \xi_i)$ of $H^{2n+\ast}$. Indeed, since $\xi_a \in \mathfrak{D}^\perp$, the $\eta_a$ vanish on $\mathfrak{D}^\perp$. Thus $\omega \circ L = 0$.

5.- THE CHEN CLASS OF A CAUCHY-RIEMANN SUBMANIFOLD.

Let M be a C.R. submanifold of $\mathbb{C}^n$. Let $\pi : N \rightarrow M$ be a $T^2$- fibration, as in theor. B. Assume s is even. Then N is a C.R.submanifold of $H^{2n+\ast}$ and its totally-real distribution is integrable. We shall need the following:

LEMMA

The holomorphic distribution of N is minimal.

Proof.

Note that we may not use lemma 4. in [17] (p.169) since its proof makes essential use of the Kaehler property. Neither could one use corollary 2.3 of [27] (p.291), (although $\mathfrak{D}^\perp_N \subseteq \mathcal{F}$) since $(\mathfrak{D}, \emptyset)$ fails to be locally conformal Kaehler. Now (2.4) - (2.5), (3.1) lead to:

$$\langle D_X fY, \xi \rangle = \frac{1}{2} \{ [\mathfrak{F}(X, Y) - \eta_a(X) \eta_b(Y)] \xi - [X - \eta_a(X) \xi_a] \eta_b(Y) \} - \frac{1}{4} \{ E(X, Y) B + \omega(Y) fX \} \tag{5.1}$$
where \( \eta = \sum \eta_s \), \( \xi = \eta^+ \). Let \( X \in \mathcal{D}_N \), \( Z \in \mathcal{D}_N^\perp \). Using (5.1) we have:

\[
(Z, \mathcal{D}_X X) = 0 \quad (Z, \mathcal{D}_X Z) = \mathcal{D}_X X = 0 \quad (W, \mathcal{D}_X X) = 0 \quad (W, \mathcal{D}_X Z) = 0
\]

Thus: \( \mathcal{D}(Z, \mathcal{D}_X X) = 0 \) and \( \mathcal{D}^\perp_N \) follows to be minimal. Let \( p = \dim_X \mathcal{D}_N \) follows to be minimal. Let \( \{X_\alpha : 1 \leq \alpha \leq 2p\} \) be a real orthonormal frame of \( \mathcal{D}_N \), where \( X_{i+p} = \mathcal{I}_X \), \( 1 \leq i \leq p \). Then \( \{X^\alpha, \xi^\alpha \} \) is an orthonormal frame of \( \mathcal{D}_N \). Let \( \lambda^\alpha, 1 \leq \alpha \leq 2p \), be differential 1-forms on \( N \) defined by \( \lambda^\alpha(X_\beta) = \delta^\alpha_\beta \), \( \lambda^\alpha(Y) = 0 \), for any \( Y \in \mathcal{D}_N \). Let \( \lambda = \lambda^1 \wedge \ldots \lambda^{2p} \wedge \eta^1 \wedge \ldots \eta^m \). Then \( \lambda \) is a globally defined \((2p+s)\)-form on \( N \), as \( \mathcal{D}_N \) is orientable. We leave it as an exercise to the reader to follow the ideas in [17] (p.170) and show that since \( \mathcal{D}_N \) is minimal and \( \mathcal{D}^\perp_N \) integrable the \((2p+s)\)-form \( \lambda \) is closed. Thus \( \lambda \) determines a cohomology class \( c(N) = [\lambda] \in H^{2p+s}(N; \mathbb{R}) \) referred to as the Chen class of \( N \).

To prove theor. B suppose \( M \) is a C.R. product, i.e. \( M \) is locally a product of a complex submanifold and a totally-real submanifold of \( \mathbb{C}P^n \), see e.g. [28], (p.63). Now C.R. products have an integrable holomorphic distribution and a minimal totally-real distribution. By (2.8), for any tangent vector fields \( X, Y \) on \( \mathbb{C}P^n \) one has:

\[
[X, Y] = [X, Y] - \alpha^s \mathbb{F}(X^H, Y^H) \xi,
\]

Then (5.2) used for \( X = X_A^-, Y = X_B^- \) leads to \( [X^H_A, X^H_B] \in \mathcal{D}_N \). Next, as \( \mathcal{D}_N \)

\[X^H_A = 0 \text{ one has}

\[
\mathcal{D}^\perp_N [X^H_A, \xi] = (D^\xi A) X^H_A = \mathcal{D}_X X^H_A \xi.
\]

We need the following:

**Lemma**

The covariant derivative \( (D_X \mathcal{D}_N) Y = D_X \mathcal{D}_N Y - \mathcal{D}_X \mathcal{D}_N Y \) of \( \mathcal{D}_N \) is expressed by:

\[
(D_X \mathcal{D}_N) Y = - h(X, \mathcal{D}_N Y) + f h(X, Y) - \frac{1}{4} \omega(Y) F X
\]

for any tangent vector fields \( X, Y \) on \( N \). Here \( f V = \text{nor}(\mathcal{D}_N V) \) for any cross-section \( V \) in \( T(N)^m \).\( N \).

**Proof**

Let also \( V = \tan(\mathcal{D}_N V) \). Using the Gauss and Weingarten formulae of \( N \) in \( H^{2n+s} \) one has:

\[
(D_X \mathcal{D}_N) Y = (D_X \mathcal{D}_N) Y + h(X, \mathcal{D}_N Y) - \frac{1}{4} \omega(Y) F X
\]

Let us use (5.1) to substitute in (5.5); a comparisson between the normal components in (5.5) leads to (5.4), Q.E.D.

Now we may use the above lemma to end the proof of the involutivity of \( \mathcal{D}_N \). Indeed, by (5.4) and (2.4) our (5.3) turns into:

\[
\mathcal{D}^\perp_N [X^H_A, \xi] = - h(\xi, \mathcal{D}_N X^H_A) + f h(\xi, X^H_A) - \frac{1}{4} \omega(X^H_A) F \xi + \frac{1}{4} \alpha^s \mathcal{D}_N X^H_A
\]

and by (2.15) one obtaines \( \mathcal{D}^\perp_N [X^H_A, \xi] = 0 \).

The last step is to establish minimality of \( \mathcal{D}^\perp_N \). Let \( q = \dim \mathcal{D}_N^\perp, X \in M \).
If \( \{ E_i : 1 \leq i \leq q \} \) is an orthonormal frame of \( \mathcal{D}^\perp \) then (2.8) yields:

\[
\langle N \sum_{i=1}^q \mathcal{D} s^i E_i \rangle_i = \langle \sum_{i=1}^q \nabla E_i \rangle_i .
\]

(5.7)

But \( \mathcal{D}^\perp \) is minimal, so the right hand member of (5.7) is zero. Finally, one may follow the ideas in [17], (p.170) to show that since \( \mathcal{D} N \) is integrable and \( \mathcal{D} N \) minimal the \((2p+s)\)-form \( \lambda \) is coclosed. As \( N \) is compact, \( \lambda \) is harmonic. Thus \( c(N) = [\lambda] \neq 0 \), and our theor. B is completely proved.

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