UNIFORM TOEPLIZT MATRICES

I. J. MADDOX
Department of Pure Mathematics
Queen's University of Belfast
Belfast BT7 1NN
Northern Ireland
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ABSTRACT. We characterize all infinite matrices of bounded linear operators on a Banach space which preserve the limits of uniformly convergent sequences defined on an infinite set. Also, we give a Tauberian theorem for uniform summability by the Kuttner-Maddox matrix.

KEY WORDS AND PHRASES. Uniform Toeplitz matrix, strong slow oscillation, Tauberian theorems.

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1. INTRODUCTION.

By T we denote any infinite set of objects and we consider functions $f_k : T \to X$ for $k = 1, 2, \ldots$, where $(X, ||.||)$ is a Banach space.

The notation $f_k \Rightarrow f$ will be used to signify that $f_k \to f$ as $k \to \infty$, uniformly on $T$, which is to say that there exists $f : T \to X$ such that for all $\epsilon > 0$ there exists $k_0 = k_0(\epsilon) > 0$ with

$$||f_k(t) - f(t)|| < \epsilon, \text{ for all } k > k_0 \text{ and all } t \in T.$$

Now suppose that for $n, k = 1, 2, \ldots$ each $A_{nk} \in B(X)$, i.e. each $A_{nk}$ is a bounded linear operator on $X$. Then we shall say that $A = (A_{nk})$ is a uniform Toeplitz matrix of operators if and only if:

$$\sum_{k=1}^{\infty} A_{nk} f_k(t) \text{ converges in the norm of } X$$

for each $n \in \mathbb{N} = \{1, 2, 3, 4, \ldots\}$ and each $t \in T$ and

$$\sum_{k=1}^{\infty} A_{nk} f_k \Rightarrow f$$

whenever $f_k \Rightarrow f$.

Following Robinson [1] and Lorentz and Macphail [2], if $(B_k)$ is a sequence in $B(X)$ we denote the group norm of $(B_k)$ by
where the supremum is over all \( p \in \mathbb{N} \) and all \( x_k \) in the closed unit sphere of \( X \).

By \( C \) we shall denote the \((C,1)\) matrix of arithmetic means, given by

\[
\begin{array}{cccccc}
1 & 0 & 0 & 0 & . & . \\
1 & 1 & 2 & 0 & 0 & . \\
1 & 1 & 1 & 0 & 0 & . \\
. & . & . & & . & . \\
. & . & . & & . & . \\
\end{array}
\]

By \( D \) we denote the Kuttner-Maddox matrix, used extensively in the theory of strong summability [3, 4, 5]:

\[
\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 0 & 0 & 0 \\
0 & 0 & 1 & 4 & 0 & 0 \\
. & . & . & & . & . \\
. & . & . & & . & . \\
\end{array}
\]

In work on strong summability it is often advantageous to use the fact that, for non-negative \( (p_k) \) the summability methods \( C \) and \( D \) are equivalent, in the sense that \( p_k \to 0(C) \) if and only if \( p_k \to 0(D) \).

In connection with Tauberian theorems we now introduce the idea of uniform strong slow oscillation.

Let \( s_k : T \to X \) for each \( k \in \mathbb{N} \). Then we say that \( (s_k) \) has uniform strong slow oscillation if and only if \( s_n - s_k \to 0 \) whenever \( k \to \infty \) and \( n > k \) with \( n/k = O(1) \).

In what follows we shall regard \( s_k \) as the \( k \)-th partial sum of a given series of functions \( \Sigma a_k = a_1 + a_2 + \ldots \), each \( a_k : T \to X \).

2. UNIFORM TOEPLITZ MATRICES.

The following theorem characterizes the uniform Toeplitz matrices of operators which were defined in Section 1.

**THEOREM 1.** \( A = (A_{nk}) \) is a uniform Toeplitz matrix if and only if

\[
\sup_n \left\| (A_{n1}, A_{n2}, \ldots) \right\| < \infty , \tag{2.1}
\]

A is column-finite, \( \tag{2.2} \)

for each \( n \in \mathbb{N} \), \( A_n : = \sum_{k=1}^{\infty} A_{nk} \) converges, \( \tag{2.3} \)

\( A_n = I \), ultimately in \( n \). \( \tag{2.4} \)

**PROOF.** We remark that in (2.3) the convergence is in the strong operator topology, and in (2.4), \( I \) is the identity operator on \( X \).

For the sufficiency, let \( H \) denote the value of the supremum in (2.1), let
n \in N \text{ and } t \in T. \text{ Then, for any } \epsilon > 0 \text{ there exists } k_0 \text{ such that }
\left| |f_k(t) - f(t)| \right| < \epsilon \text{ for all } k > k_0.

Now for each } p \in N,
\sum_{k=1}^{p} A_{nk} (f_k(t) - f(t)) + \sum_{k=p}^{p} A_{nk} f(t),

where we assume that \( f_k \Rightarrow f \). \text{ By (2.3), as } p \to \infty \text{, we have }
\sum_{k=1}^{p} A_{nk} f(t) \to A_{n} f(t).

Also, if } s > r > k_0,
\left| \sum_{k=r}^{s} A_{nk} (f_k(t) - f(t)) \right| \leq H \epsilon,

whence \( \sum_{k=1}^{s} A_{nk} f(t) \) converges.

By (2.4) there exists } m \in N \text{ such that } A_n = I \text{ for all } n > m, \text{ and by (2.2)}
\text{ there exists } n_0(\epsilon) \in N \text{ such that } A_{nk} = 0 \text{ for } 1 \leq k \leq k_0 \text{ and for } n > n_0(\epsilon).

Taking } n > m + n_0 \text{ we have }
\sum_{k=1}^{\infty} A_{nk} f_k(t) = f(t) + \sum_{k=1}^{\infty} A_{nk} (f_k(t) - f(t)).

Since
\left| \sum_{k=1}^{\infty} A_{nk} (f_k(t) - f(t)) \right| \leq \epsilon \left( |A_{n1}, A_{n2}, \ldots| \right),

it follows by (2.1) that \( \sum_{nk} A_{nk} f_k \Rightarrow f \), which proves the sufficiency.

Now consider the necessity. \text{ Take any convergent sequence } (x_k) \text{ in } X, \text{ with } x_k \to x. \text{ Define } f_k(t) = x_k \text{ for all } k \in N \text{ and all } t \in T, \text{ and define } f(t) = x \text{ for all } t \in T. \text{ Then } f_k \Rightarrow f \text{ and so } \sum_{nk} A_{nk} f_k \text{ converges for each } n \text{ and tends to } x, \text{ whence the usual Toeplitz theorem for operators, see Robinson [1] or Maddox [6],}
\text{ yields (2.1) and (2.3) of our present theorem.}

Next, suppose that (2.4) is false. \text{ Then there exist natural numbers } n(1) < n(2) < \ldots \text{ with } A_{n(i)} \neq I \text{ for all } i \in N. \text{ Hence there exist } x_i, x_{i-1} \in X \text{ with }
\left| |A_{n(i)} x_i - x_{i-1}| \right| > 0 \quad (2.5)

for all } i \in N. \text{ Let us write } y(i) \text{ for the expression inside the norm bars in (2.5).}
\text{ Since } T \text{ is an infinite set we may choose any countably infinite subset }
\{t_1, t_2, t_3, \ldots \} \text{ of } T. \text{ Then we define } f : T \to X \text{ by }
f(t_i) = x_i / |y(i)| \quad (2.6)
for all \( i \in \mathbb{N} \), and \( f(t) = 0 \) otherwise. If we define \( f_k = f \) for all \( k \in \mathbb{N} \) then we certainly have \( f_k \to f \). But \( A \) is not a uniform Toeplitz matrix, since for \( n = n(i) \) we have by (2.6),

\[
\left| \sum_{k=1}^{n} A_{nk} f(t_i) - f(t_i) \right| = \left| \sum_{x=1}^{n} x_i - x_i \right| = 1.
\]

Hence, if \( A \) is a uniform Toeplitz matrix then (2.4) must hold, and a similar argument shows that (2.2) is necessary, which completes the proof of the theorem.

Since \( C \), the \((C,1)\) matrix, is not column-finite we immediately obtain:

**COROLLARY 2.** \( C \) is a Toeplitz matrix but not a uniform Toeplitz matrix.

However, since the elements of the Kuttner-Maddox matrix \( D \) are non-negative and its row sums all equal 1 it is clear that the conditions of Theorem 1 hold, whence \( D \) is a uniform Toeplitz matrix. Thus, whenever \( f_k \to f \) it follows that

\[
2^{-r} \sum_{k=r}^{2r} f_k \to f,
\]

where the sum in (2.7) is over \( 2^r \leq k < 2^{r+1} \) for \( r = 0,1,2,\ldots \). We also express (2.7) by writing \( f_k \to f(D) \).

The relation between \( C \) and \( D \) for uniform summability is given by:

**THEOREM 3.** \( f_k \to f(C) \) implies \( f_k \to f(D) \), but not conversely in general.

**PROOF.** Write

\[
c(n) = n^{-1} \sum_{k=1}^{n} f_k(t) \quad \text{and} \quad d(r) = 2^{-r} \sum_{k=r}^{2r} f_k(t).
\]

Then we find that

\[
d(r) = (2 - 2^{-r}) c(2^{r+1} - 1) - (1 - 2^{-r}) c(2^r - 1),
\]

and it is clear that the right-hand side of (2.8) defines a uniform Toeplitz transformation between the \( c \) and \( d \) sequences.

For the last part of the theorem we may define real-valued functions on \( T \) by

\[
f_k(t) = 2^r \quad \text{when} \quad k = 2^r \quad \text{and} \quad f_k(t) = -2^r \quad \text{when} \quad k = 1 + 2^r, \quad \text{and} \quad f_k(t) = 0 \quad \text{otherwise}.
\]

Then \( f_k \to 0(D) \). Now suppose, if possible, that \( f_k \to f(C) \), which implies \( f_k \to f(D) \). Hence \( f = 0 \). But

\[
c(2^r) - (1 - 2^{-r}) c(2^r - 1) = 1,
\]

contrary to the fact that \( c(n) \to 0 \).

3. **A UNIFORM TAUBERIAN THEOREM.**

By the remark following Corollary 2 we know that \( f_k \to f \) implies \( f_k \to f(D) \), but the example of Theorem 3 shows that the converse is generally false. The next result shows that uniform strong slow oscillation is a Tauberian condition for uniform \( D \) summability.
THEOREM 4. If \((s_k)\) has uniform strong slow oscillation and \(s_k \Rightarrow f(D)\) then \(s_k \Rightarrow f\).

PROOF. Without loss of generality we may suppose that \(f = 0\).

Take \(n \in \mathbb{N}\) and determine \(r\) such that \(2^r \leq n < 2^{r+1}\). If \(e > 0\) there exists \(r_0\) such that if \(2^r \leq k < 2^{r+1}\) then

\[
||s_k(t) - s_n(t)|| < e
\]

whenever \(r > r_0\) and \(t \in T\). Since

\[
2^{-r} \sum_{k=1}^{2^r} s_k(t) = s_n(t) + 2^{-r} \sum_{k=1}^{2^r} (s_k(t) - s_n(t))
\]

we see that \(s_n \Rightarrow 0\).

Our final result shows that the natural conditions \(kak \Rightarrow 0\) or \(kak \Rightarrow O(C,1)\) are both Tauberian conditions for uniform \(D\) summability, but that the restriction \(kak \Rightarrow 0\) cannot be relaxed to the uniform boundedness of \((kak)\).

THEOREM 5. (i) If \(kak \Rightarrow 0\) or \(kak \Rightarrow O(C,1)\) and \(s_k \Rightarrow f(D)\) then \(s_k \Rightarrow f\).

(ii) There exists a divergent series \(\Sigma a_k\) with \((kak)\) uniformly bounded and \(s_k \Rightarrow O(D)\).

PROOF. (i) First note that \(kak \Rightarrow 0\) does not generally imply \(kak \Rightarrow O(C,1)\) because \((C,1)\) is not a uniform Toeplitz matrix by Corollary 2. We shall show that \(kak \Rightarrow O(C,1)\) is a Tauberian condition for \(D\), the proof for \(kak \Rightarrow 0\) being similar. In fact we shall show that \(kak \Rightarrow O(C,1)\) implies that \((s_k)\) has uniform strong slow oscillation.

Let us write \(a_k = a_k(t)\), \(s_n = s_n(t)\) and

\[
A_n = n^{-1} \sum_{k=1}^{n} ka_k,
\]

with the assumption that \(A \Rightarrow 0\). Then for \(n > k \geq 1\), by partial summation,

\[
s_n - s_k = A_n - \sum_{k=1}^{n} \frac{A_k}{k} + \sum_{\nu=k+1}^{n-1} (\nu a_{\nu} - k a_k) / (\nu + 1),
\]

whence

\[
||s_n - s_k|| \leq \max(||A\nu|| : k \leq \nu \leq n)(1 + \sum_{\nu=k+1}^{n} \frac{1}{\nu + 1}).
\]

If \(n/k = O(1)\) then

\[
1 + \frac{k}{n} + 2 \sum_{\nu=k+1}^{n} \frac{1}{\nu + 1} < 2 + 2\frac{n}{k} = O(1),
\]

and so \(s_n - s_k \Rightarrow 0\), as required.

(ii) Define a numerical sequence \((s_k)\) by \(s_k = 0\) when \(1 \leq k < 4\), and for \(n \geq 2\)
define \( s_k = 0 \) when \( k = 2^n \) and when \( k = 2^n + 2 \times 2^{n-2} \); \( s_k = 1 \) when \( k = 2^n + 2^{n-2} \) and \( s_k = -1 \) when \( k = 2^n + 3 \times 2^{n-2} \). Otherwise define \( s_k \) linearly, so that the graph of \( (s_k) \) is a triangular-shaped wave. Then \((s_k)\) diverges and it is clear that \( \sum r s_k = 0 \) for all \( r \geq 0 \). Also, it is easy to check that \( k|s_k| \leq 8 \) for all \( k \geq 1 \), whence our result follows on defining \( s_k(t) = s_k \) for all \( k \geq 1 \) and all \( t \in T \).

REFERENCES


