A NOTE ON SOME SPACES $L_{\gamma}$ OF DISTRIBUTIONS WITH LAPLACE TRANSFORM

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(Received March 2, 1989)

ABSTRACT. In this paper we calculate the dual of the spaces of distributions $L_{\gamma}$ introduced in [1]. Then we prove that $L_{\gamma}$ is the dual of a subspace of $C_c^\infty(\mathbb{R})$.

KEY WORDS AND PHRASES. Convolution, Laplace Transform, Strict Inductive Limit.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODE. Primary 44A35, Secondary 44A10

1. INTRODUCTION

Let $D'$ and $S'$ be the classical Schwartz's spaces of distributions in $\mathbb{R}$ and denote by $L$ the Laplace transformation. In (Pérez-Esteva [1]) were introduced spaces $L_a^\gamma$ as follows:

$L_{\gamma}^a$ is the subspace of $L^1_{\text{loc}}(\mathbb{R})$ of functions $f$ with supp $f \subset [a,\infty)$ and $e^{-\gamma} f \in L^2(\mathbb{R})$, where $e^{-\gamma}(x)=e^{-\gamma x}$. $L_{\gamma}^a$ is a Hilbert space with the inner product

$$(f,g) = \int e^{-2\gamma} f^* g \, dx$$

then we define $L_{\gamma}^a = p_{\gamma}^{D^a}$ where $D^p$ is the distributional derivative of order $p$.

Since $p_{\gamma}^{D^a} \subset L_{\gamma}^a$ is bijective, we can copy the Hilbert space structure of $L_{\gamma}^a$ on $L_{\gamma}^a$. We have the continuous inclusions

$L_{\gamma}^a \subset L_{\gamma}^{b}$, for $a > b$

$L_{\gamma}^a \subset L_{\gamma}^{b}$, if $p < p'$. Hence for $p = \{0,1,\ldots\}$ the strict inductive limit

$L_{\gamma} = \text{ind lim}_{a \to -\infty} L_{\gamma}^a$

makes sense. Then

$L_{\gamma} = \text{ind lim}_{p \to \infty} L_{\gamma} = \text{ind lim}_{p \to \infty} L_{\gamma}^{-p}$

is also well defined.

In [1] it was studied the spaces of distributions $g$ for which the convolution

$f \ast fg : L_{\gamma} \to L_{\gamma}$

is continuous.
Here we describe the strong dual of $L^\gamma$, which turns out to be a subspace $S^\gamma$ of $C^\infty(\mathbb{R})$. Then we prove the reflexivity of $S^\gamma$ and conclude that $(S^\gamma)' = L^\gamma$, which is the main result of the paper. $\| \cdot \|_2$ will denote the norm of $L^2(\mathbb{R})$, $\gamma$ will be assumed to be a positive constant, and $N$ will be the set of nonnegative integers.

2. THE DUAL OF $L^\gamma$

DEFINITION 1. Let $L^\gamma$ be the space of all complex measurable functions $g$ in $\mathbb{R}$ such that $\chi_{(a,\infty)} e^{-\gamma g} \in L^2(\mathbb{R})$ for every $a \in \mathbb{R}$, where $\chi_{(a,\infty)}$ stands for the characteristic function of $(a,\infty)$. We provide $L^\gamma$ with the topology given by the seminorms

$$p_a(g) = \| \chi_{(a,\infty)} e^{-\gamma g} \|_2, \quad a \in \mathbb{R}.$$ 

Next we denote by $S^\gamma$ the subspace of $L^\gamma$ such that $D^n g \in L^\gamma$ for every $n \in \mathbb{N}$. Define the topology of $S^\gamma$ by the system of seminorms

$$p_n(g) = \| \chi_{(a,\infty)} e^{-\gamma D^n g} \|_2, \quad a \in \mathbb{R}, \quad n \in \mathbb{N}.$$ 

It is clear that $L^\gamma$ and $S^\gamma$ are Frechet spaces and since $D^n g \in L^1_{\text{loc}}(\mathbb{R})$ for any $n \in \mathbb{N}$ and $g \in S^\gamma$, we have that $S^\gamma \subset C^\infty(\mathbb{R})$.

LEMMA 1. Let $\phi \in L^\gamma'$, then for every $p \in \mathbb{N}$, there exists $g_p \in L^\gamma$ such that

$$\phi(D^p f) = \int_{\mathbb{R}} e^{-2\gamma f} g_p dx, \quad f \in L^\gamma_{\Theta^y}$$

The sequence $\{g_p\}_{p \in \mathbb{N}}$ satisfies

$$g_{p+1} = -Dg_p + 2\gamma g_p, \quad p \in \mathbb{N} \tag{2.1}$$

Hence $\phi$ is determined by $g_0 \in S^\gamma$.

PROOF. Fix $a \in \mathbb{R}$ and $p \in \mathbb{N}$. Then $\phi \in (L^a_{\Theta^y})'$, and there exists $g_{pa} \in L^a_{\Theta^y}$ such that

$$(D^p f) = \int_{\mathbb{R}} e^{-2\gamma f} g_{pa} dx, \quad D^p f \in L^a_{\Theta^y}$$

If $a < b$, we have $L^b_{\Theta^y} \subset L^a_{\Theta^y}$, then

$$\phi(D^p f) = \int_{\mathbb{R}} e^{-2\gamma f} g_{pb} dx = \int_{\mathbb{R}} e^{-2\gamma f} \chi_{(b,\infty)} g_{pa} dx$$

for $D^p f \in L^b_{\Theta^y}$, which shows that

$$g_{pb} = \chi_{(b,\infty)} g_{pa}$$

If $g_{pa}$ is the restriction of $g_{pa}$ to $(a,\infty)$, then $g_p = \bigcup_a g_{pa}$ is well defined, belongs to $L^\gamma$ and

$$\phi(D^p f) = \int_{\mathbb{R}} e^{-2\gamma f} g_{p} dx, \quad D^p f \in L^\gamma_{\Theta^y}$$

Let $\varphi \in \mathcal{D}$. Since $D^{p+1}\varphi \in L^{p+1}_{\Theta^y} \cap L^\gamma_{\Theta^y}$, we have
\[ \phi(p^{+1}\varphi) = \int e^{-2\gamma p + 1} \varphi g_p \, dx = \int e^{-2\gamma D} \varphi g_p \, dx \]
\[ = \int [D(e^{-2\gamma \varphi}) + 2\gamma e^{-2\gamma \varphi}] g_p \, dx \]
\[ = <e^{-2\gamma D} g_p + 2\gamma e^{-2\gamma g_p}, \varphi> \]

where \(<,\rangle\) represents the duality between \(D\) and \(D'\). It follows that

\[ g_{p+1} = -D g_p + 2\gamma g_p \]
or

\[ e^{-2\gamma} g_{p+1} = -D(e^{-2\gamma} g_p) \]

Hence, every \(g_p\) belongs to \(S_\gamma\).

**Lemma 2.** Let \(g \in S_\gamma\) and \(H\) be the differential operator defined by \(H = -D + 2\gamma I\). Then the functional

\[ \phi(D^p f) = \int e^{-2\gamma f} H(p) g dx, \quad f \in \mathcal{L}_{\mathcal{O}_\gamma} \]

is well defined in \(\mathcal{L}\) and is continuous.

**Proof.** Let \(f \in \mathcal{L}^a_{\mathcal{O}_\gamma}\) be such that \(f = Dh\) with \(h \in \mathcal{L}_{\mathcal{O}_\gamma}\). There exists a sequence \(\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{D}\) converging to \(f\) in \(L^b_{\mathcal{O}_\gamma}\) if \(b < a\).

Let

\[ \varphi_n(x) = \int_{-\infty}^{x} f_n \, dy \]

Then \(f \in L^b_{\mathcal{O}_\gamma}\), \(D(\varphi_n - h) = f_n - f\), and since the inclusion \(L^b_{\mathcal{O}_\gamma} \subset \mathcal{L}^b_{\mathcal{O}_\gamma}\) is continuous, we have that \(\{\varphi_n\}_{n \in \mathbb{N}}\) converges to \(h\) in \(\mathcal{L}_{\mathcal{O}_\gamma}\). It follows that

\[ \int e^{-2\gamma h} H(g) dx = \lim_{n \to \infty} \int e^{-2\gamma \varphi_n} H(g) dx \quad (2.2) \]

and

\[ \int e^{-2\gamma f_n g} dx = \lim_{n \to \infty} \int e^{-2\gamma f_n g} dx \quad (2.3) \]

On the other hand

\[ \int_{-\infty}^{B} e^{-2\gamma \varphi_n} H(g) dx = \int_{-\infty}^{B} \varphi_n D(e^{-2\gamma g}) dx \]
\[ = -\varphi_n(B)e^{-2\gamma(B)}g(B) + \int_{-\infty}^{B} f_n e^{-2\gamma g} dx \quad (2.4) \]

But we have the estimate

\[ |g(x)| \leq |g(b)| + e^{\gamma(x)} \|\chi_{[b,\omega]} e^{-\gamma(Dg - \gamma g)}\|_2 (x-b)^{1/2} \quad \text{for } x > b \]

Hence

\[ \int e^{-2\gamma \varphi_n} H(g) dx = \int e^{-2\gamma f_n g} dx \]

From (2.2) and (2.3) it follows that

\[ \int e^{-2\gamma f_n g} dx = \int e^{-2\gamma h} H(g) dx \quad (2.5) \]
By induction we obtain
\[ \int e^{-2\gamma} f \, g \, dx = \int e^{-2\gamma} h \, H^p(g) \, dx \quad (2.6) \]
if \( f = D^p h \) and \( f, h \in L_{oY} \).

Finally, if \( D^p f = D^q h \) with \( f, h \in L_{oY} \) and \( q \geq p \), then \( f = D^{q-p} h \), hence by (2.6) we have
\[ \int e^{-2\gamma} f \, H^p(g) \, dx = \int e^{-2\gamma} h \, H^q(g) \, dx \]
Thus \( \Phi \) is well defined and it is clearly continuous.

**THEOREM 1.** The strong dual of \( L_{aY} \) is \( S_{aY} \).

**PROOF.** By lemmas and 2 we know that \( L_{aY}^* = S_{aY} \). It remains to prove that the strong topology \( \mathcal{A}(L_{aY}^*, L_{aY}) \) coincides with the topology \( \tau \) of \( S_{aY} \). First notice that \( \tau \) is defined by the system of seminorms
\[ q_{ap}(g) = \| x(a, \omega) e^{-\gamma} H^p(g) \|_2, \quad a \in \mathbb{R}, \quad p \in \mathbb{N} \]
Fix \( a \in \mathbb{R} \) and \( p \in \mathbb{N} \). Let \( V = \{ g \in S_{aY} : q_{ap}(g) \leq 1 \} \). Denote by \( U \) the unit ball in \( f_{oY}^* \), then the set \( B = D^p U \) is bounded in \( L_{oY}^* \) and hence in \( L_{oY} \). If \( g \in B^0 \) (the polar of \( B \)), then for every \( f \in U \) we have
\[ \| e^{-2\gamma} f \, H^p(g) \|_2 \leq 1 \]
Thus
\[ \| e^{-2\gamma} x(a, \omega) \, H^p(g) \|_2 \leq 1 \]
It follows that \( B^0 \subset V \) and \( \tau \subset \beta(L_{oY}^*, L_{oY}) \). Now, let \( B \) be a bounded set in \( L_{oY} \). Then for some \( p \in \mathbb{N}, B \subset L_{oY}^p \) and is bounded there (see Kucera, McKennon [2]). Hence \( B \subset D^p U \) for some \( p \in \mathbb{N} \) and is bounded there (see Kucera, McKennon [2]).

**COROLLARY 1.** \( L_{aY} \) is the strong dual of \( S_{aY} \).

**PROOF.** By (Kucera, McKennon [2], Theorem 4) we know that \( L_{aY} \) is reflexive. Hence the corollary follows from Theorem 1.

**REFERENCES**


