ON THE δ-CONTINUOUS FIXED POINT PROPERTY

F. CAMMAROTO and T. NOIRI

Dipartimento di Matematica
Università di Messina
98166 Sant'Agata (Messina)
ITALY

Department of Mathematics
Yatsushiro College of Technology
Yatsushiro, Kumamoto
866 JAPAN

(Received July 7, 1988 and in revised form November 6, 1988)

ABSTRACT. In this paper, we define and investigate the δ-continuous retraction and the δ-continuous fixed point property. Theorem 1 of Connell [11] and Theorem 3.4 of Arya and Deb [2] are improved.

KEY WORDS AND PHRASES. δ-continuous, θ-continuous, weakly-continuous, semi-regular, almost-regular, fixed point property.
1980 MATHEMATICS SUBJECT CLASSIFICATION CODE. 54 C 10, 54 C 20.

0. INTRODUCTION.

The notion of θ-continuous functions was first introduced by Fomin [1]. After that, this notion has been widely investigated in the literature. By utilizing θ-continuous functions, Arya and Deb [2] defined and investigated the θ-continuous retraction, the θ-continuous fixed point property and the θ-continuous homotopy. On the other hand, in [3] and [4] the present authors have independently introduced the notion of δ-continuous functions. The purpose of this paper is to apply δ-continuity to the retraction and the fixed point property. In Section 2, we study the retraction of a topological space by δ-continuous functions. Section 3 deals with the fixed point property in relation to δ-continuous functions. The main results of this paper are Theorems 3.2 and 3.3 which improve Theorem 1 of [11] and Theorem 3.4 of [2], respectively.
1. PRELIMINARIES.

Throughout the present paper, spaces will always mean topological spaces on which no separation axioms are assumed unless explicitly stated. We shall denote a topological space by \((X, \tau)\) or simply by \(X\). Let \((X, \tau)\) be a space and \(A\) a subset of \(X\). The closure of \(A\) and the interior of \(A\) are denoted by \(\overline{A}\) and \(\overset{i}{A}\) (or simply \(A\) and \(A\)), respectively. A subset \(A\) of \(X\) is said to be regular open (resp. regular closed) if \(A=(\overset{i}{A})^c\) (resp. \(A=\overline{A}\)). The family of regular open sets of \(X\) will be denoted by \(\text{RO}(X)\). A point \(x\) of \(X\) is said to be in the \(\delta\)-closure \([5]\) of \(A\), denoted by \(\text{Cl}_{\delta}(A)\), if \(A \cap V \neq \emptyset\) for every \(V \in \text{RO}(X)\) containing \(x\). A subset \(A\) is said to be \(\delta\)-closed \([5]\) if \(A=\text{Cl}_{\delta}(A)\). The complement of a \(\delta\)-closed set is said to be \(\delta\)-open. The topology on \(X\) which has \(\text{RO}(X)\) as a basis is called the semi-regularization of \(\tau\) and is denoted by \(\tau^*\). It is obvious that every element of \(\tau^*\) is a \(\delta\)-open set of \((X, \tau)\). A space \((X, \tau)\) is said to be semi-regular if \(\tau = \tau^*\). A space \((X, \tau)\) is said to be almost-regular \([6]\) if for each regular closed set \(F\) and each \(x \in X - F\), there exist open sets \(U\) and \(V\) such that \(x \in U\), \(F \subseteq V\) and \(U \cap V = \emptyset\).

DEFINITION 1.1. A function \(f : (X, \tau) \rightarrow (Y, \sigma)\) is said to be \(\delta\)-continuous \([3, 4]\) (resp. almost-continuous \([7]\), \(\theta\)-continuous \([1]\) and weakly continuous \([8]\)) if for each \(x \in X\) and each open neighborhood \(V\) of \(f(x)\), there exists an open neighborhood \(U\) of \(x\) such that \(f(U) \subseteq V\) (resp. \(f(U) \subseteq V\) and \(f(U) \subseteq V\)).

REMARK 1.1. It is shown in \([2, 3, 9]\) that the following implications hold: \(\delta\)-continuous \(\Rightarrow\) almost-continuity \(\Rightarrow\) \(\theta\)-continuous \(\Rightarrow\) weak-continuity, where none of these implications is reversible.

2. \(\delta\)-CONTINUOUS RETRACTIONS.

Arya and Deb \([2]\) defined a subset \(A\) of a space \(X\) to be a \(\theta\)-continuous retract of \(X\) if there exists a \(\theta\)-continuous function \(f : X \rightarrow A\) such that \(f/A\) is the identity on \(A\). We shall similarly define a \(\delta\)-continuous retract.

DEFINITION 2.1. A subset \(A\) of space \(X\) is called a \(\delta\)-continuous retract of \(X\) if there exists a \(\delta\)-continuous function \(f : X \rightarrow A\) such that \(f\) is the identity on \(A\), that is, \(f(x) = x\) for every \(x \in A\). And such a function \(f\) is called a \(\delta\)-continuous retraction.

REMARK 2.1. It is obvious that every \(\delta\)-continuous retract is a \(\theta\)-continuous retract. However, every \(\delta\)-continuous retract is not necessarily a \(\theta\)-continuous retract as the following example shows.

EXAMPLE 2.1. Let \(X = \{a, b, c, d\}\) and \(\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}\). Let \(A = \{a, b, c\}\) and \(f : (X, \tau) \rightarrow (A, \tau / A)\) be a function defined as follows: \(f(a) = a, f(b) = b, f(c) = c\) and \(f(d) = d\). Then \(A\) is a \(\delta\)-continuous retract of \(X\) but it is not a continuous retract of \(X\) since \(f^{-1}(\{a\}) \neq \tau\) for \(\{a\} \in \tau / A\).

REMARK 2.2. In Example 3.1 of \([2]\), Arya and Deb showed that every \(\theta\)-continuous retract is not necessarily a continuous retract. However, this example is false. The \(\theta\)-continuous function \(f : X \rightarrow A\) in \([2, \text{Example 3.1}]\) is necessarily continuous since the subspace \(A\) is discrete and regular. Since every \(\delta\)-continuous function is \(\theta\)-continuous, Example 2.1 also shows that every \(\theta\)-continuous retract is not a continuous retract.

We shall investigate relationships between \(\delta\)-continuous retract and continuous retract.

PROPOSITION 2.1. If \(X\) is a semi-regular space and \(A\) is a continuous retract of \(X\), then \(A\) is a \(\delta\)-continuous retract of \(X\).

PROOF. This follows from the fact that a continuous function from a semi-regular space is \(\delta\)-continuous \([3, \text{Prop. 1.5}]\).

LEMMA 2.1. If \(A\) is either open or dense in a space \(X\) and \(V \in \text{RO}(X)\), then \(V \cap A\) is regular open in the subspace \(A\).

PROOF. If \(A\) is open in \(X\), then this follows from \([10, \text{p. 175, B}]\). Next, suppose that \(A\) is open in \(X\) and \(V \in \text{RO}(X)\). Then, we have \(\overline{V \cap A} = (\overline{V} \cap A)\) (\(\overline{V} \cap A\)).

We have \(\overline{V} \cap A = (\overline{V} \cap A)\overset{i}{A} = \overline{V} \cap A\).
Moreover, we have $\overline{V \cap A} \cap A \supseteq (V \cap \overline{A})' \cap A = \overline{V \cap A}$.

Therefore, we obtain $\overline{V \cap A} = \overline{V \cap A}$ and hence $V \cap A$ is regular open in $A$.

**Proposition 2.2.** Let $X$ be a semi-regular space and $A$ either open or dense in $X$. Then $A$ is a continuous retract of $X$ if and only if $A$ is a $\delta$-continuous retract of $X$.

**Proof.** From Lemma 2.1, for $A$ either open dense and $X$ semiregular, $\tau = \tau^*$ and $(\tau / A) = (\tau^* / A) \subseteq (\tau / A)^* \subseteq (\tau / A)$.

Therefore, $A$ is semiregular so that $f : X \rightarrow A$ is $\delta$-continuous if and only if it is continuous.

**Proposition 2.3.** Let $X$ be a space and $A$ a semi-regular (resp. almost-regular) subspace of $X$. If $A$ is a $\delta$-continuous (resp. continuous) retract of $X$, then it is a continuous (resp. $\delta$-continuous) retract of $X$.

**Proof.** Let $f : X \rightarrow A$ be a $\delta$-continuous retraction and $A$ be semi-regular. Every $\delta$-continuous function into a semi-regular space is continuous [3, Prop. 1.4]. Therefore, $A$ is a continuous retract of $X$. Every continuous function into an almost regular space is $\delta$-continuous [3, Prop. 1.8]. Therefore, the second part follows.

**Theorem 2.1.** If $A$ is a $\delta$-continuous retract of $X$ and $B$ is a $\delta$-continuous retract of $A$, then $B$ is a $\delta$-continuous retract of $X$.

**Proof.** Let $f : X \rightarrow A$ and $g : A \rightarrow B$ be $\delta$-continuous retractions. The composite function $g \circ f : X \rightarrow B$ is $\delta$-continuous [3, Prop. 3.2]. Moreover, we have $(g \circ f)(x) = g(f(x)) = g(x) = x$ for every $x \in B \subseteq A$. Therefore, $g \circ f : X \rightarrow B$ is a $\delta$-continuous retraction and hence $B$ is a $\delta$-continuous retract of $X$.

**Theorem 2.2.** A subset $A$ of a space $X$ is a $\delta$-continuous retract of $X$ if and only if for every space $Y$, every $\delta$-continuous function $f : A \rightarrow Y$ can be extended to a $\delta$-continuous of $X$ into $Y$.

**Proof.** Necessity. Let $g : X \rightarrow A$ be a $\delta$-continuous retraction. Let $Y$ be any space and $f : A \rightarrow Y$ be any $\delta$-continuous function. Then composite function $f \circ g : X \rightarrow Y$ is $\delta$-continuous [3, Prop. 3.2]. Moreover, we have $(f \circ g)(x) = f(g(x)) = f(x)$ for every $x \in A$. Therefore, $f \circ g$ is an extension of $f$.

Sufficiency. Let $i_A : A \rightarrow A$ be the identity function on $A$. Then $i_A$ is $\delta$-continuous and hence by the hypothesis there exists a $\delta$-continuous function $g : X \rightarrow A$ such that $g / A = i_A$. Therefore, $A$ is a $\delta$-continuous retract of $X$.

**Theorem 2.3.** If $A$ is a $\delta$-continuous retract of a Hausdorff space $X$, then $A$ is $\delta$-closed in $X$.

**Proof.** Let $f : X \rightarrow A$ be a $\delta$-continuous retraction. Suppose that $A$ is not $\delta$-closed in $X$. There exists a point $x \in Cl_\delta(A) - A$. Since $x \notin A$, $f(x) \neq x$ and hence there exist open sets $U$ and $V$ such that $x \in U$, $f(x) \in V$ and $U \cap V = \emptyset$; hence $U \cap V' = \emptyset$. Let $W$ be any regular open set containing $x$. Then $U \cap W$ is a regular open set containing $x$. Since $x \in Cl_\delta(A)$, $[U \cap W'] \cap A \neq \emptyset$. Let $a \in [U \cap W'] \cap A$, then $f(a) = a \in U$ and hence $f(a) \notin \overline{V}$. This shows that $f(W) \nsubseteq \overline{V}$ for any regular open set $W$ containing $x$. This contradicts the fact that $f$ is $\delta$-continuous.

3. **The $\delta$-Continuous Fixed Point Property.**

Arya and Deb [2] defined a space $X$ to have the $\theta$-continuous fixed point property if, for every $\theta$-continuous function $f : X \rightarrow X$, there exists an $x \in X$ such that $f(x) = x$. We shall define the $\delta$-continuous (resp. weakly continuous) fixed point property as follows:

**Definition 3.1.** A space $X$ is said to have the $\delta$-continuous (resp. weakly continuous) fixed point property, briefly denoted by $\delta$-FPP (resp. wcFPP), if for every $\delta$-continuous (resp. weakly continuous) function $f : X \rightarrow X$, there exists an $x \in X$ such that $f(x) = x$.

**Remark 3.1.** It is obvious that a space with the wcFPP has necessarily the $\theta$-continuous fixed point property and a space with the $\theta$-continuous fixed point property has both the $\delta$-FPP and the fixed point property.
We give an example that a space with the fixed point property need not have the &cFPP.

**Example 3.1.** Let \( X = \{a, b, c\} \) and \( \tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\} \). Then the space \((X, \tau)\) has the fixed point property [2, Example 3.2]. Now, let \( f: (X, \tau) \to (X, \tau) \) be a function defined by \( f(a) = f(c) = b \) and \( f(b) = c \). Then \( f \) is \( \delta \)-continuous but does not have a fixed point. Therefore, \((X, \tau)\) does not have the &cFPP.

**Remark 3.2.** We need the following two spaces which we were unable to obtain:

1. A space which has &cFPP but does not have the fixed point property.
2. A space which has the &cFPP but does not have the wcFPP.

**Theorem 3.1.** Let \( A \) be either open or dense in a space \( X \). If \( X \) has the &cFPP and \( A \) is a \( \delta \)-continuous retract of \( X \), then \( A \) has the &cFPP.

**Proof.** Let \( f: A \to A \) be any \( \delta \)-continuous function. Since \( A \) is a \( \delta \)-continuous retract of \( X \), by Theorem 2.2 \( f \) can be extended to a \( \delta \)-continuous function \( F: X \to A \). Let \( j: A \to X \) be the inclusion. If \( V \) is a regular open set of \( X \), then \( j^{-1}(V) = A \cap V \) is regular open in the subspace \( A \) by Lemma 2.1. Therefore, \( F^{-1}(j^{-1}(V)) = (j \circ F)^{-1}(V) \) is \( \delta \)-open in \( X \) and hence \( j \circ F: X \to X \) is \( \delta \)-continuous. Since \( X \) has the &cFPP, \( x = (j \circ F)(x) = j(f(x)) = f(x) \) for some \( x \in A \subseteq X \). This shows that \( A \) has the &cFPP. The following theorem is a slight modification of Theorem 1 of [11].

**Theorem 3.2.** Let \((X, \tau)\) be an almost-regular space with the &cFPP. If \( \sigma \) is a topology for \( X \) stronger than \( \tau \) and \( (\sigma, \tau) = (\sigma, \tau) \) for every \( G \in \sigma \), then \((X, \sigma)\) has the fixed point property.

**Proof.** Suppose that \( f: (X, \sigma) \to (X, \sigma) \) is any continuous function. Let \( g: (X, \sigma) \to (X, \tau) \) and \( h: (X, \tau) \to (X, \tau) \) be the functions defined by \( g(x) = h(x) = f(x) \) for every \( x \in X \). Let \( i: (X, \tau) \to (X, \sigma) \) be the identity function. Then, since \( \tau \subseteq \sigma \), \( i \) is an open bijection. Moreover, since \( f = i \circ g \) is continuous, \( g \) is continuous. Next, we shall show that \( h \) is \( \delta \)-continuous. Let \( x \in X \) and \( h(x) \in V \subseteq RO(X, \tau) \). Since \( (X, \tau) \) is almost-regular, there exists \( G \in \tau \) such that \( h(x) \in G \subseteq C \subseteq V \). Since \( g \) is continuous, \( g^{-1}(G) \subseteq \sigma \) and \( h^{-1}(G) = f^{-1}(G) = g^{-1}(G) \subseteq \sigma \). Therefore, \( h^{-1}(G) = f^{-1}(G) \subseteq \sigma \) and hence, utilizing continuity of \( f \) we obtain \( x \in h^{-1}(G) \subseteq C \subseteq f^{-1}(G) \subseteq C \subseteq f^{-1}(G) \subseteq C \subseteq f^{-1}(G) \subseteq C \subseteq f^{-1}(G) \subseteq C \subseteq f^{-1}(G) \subseteq C \subseteq f^{-1}(V) = h^{-1}(V) \). Now, if \( U = h^{-1}(G) \subseteq V \), then we have \( x \in U \subseteq RO(X, \tau) \) and \( h(U) \subseteq V \). This shows that \( h \) is \( \delta \)-continuous. Since \((X, \tau)\) has the &cFPP, there exists \( x \in X \) such that \( x = h(x) = f(x) \). This shows that \((X, \sigma)\) has the fixed point property.

**Corollary 3.1 (Connell [11]).** Suppose \((X, \tau)\) is a regular space with the fixed point property. If \( \sigma \) is a topology for \( X \), \( \tau \subseteq \sigma \) and \( G(\sigma) = G(\tau) \) for each \( G \in \sigma \), then \((X, \sigma)\) has the fixed point property.

**Proof.** It is shown in [3, Corollary 1.8] that if \( Y \) is regular, then \( f: X \to Y \) is \( \delta \)-continuous if and only if it is continuous. Since every regular space is almost regular, this is an immediate consequence of Theorem 3.2.

We shall give a lemma which will be used in the proof of the final theorem.

**Lemma 3.1.** Let \( f: X \to Y \) and \( g: Y \to Z \) be functions:

1. \( f \) is weakly continuous if and only if \( f^{-1}(V) \subseteq f^{-1}(V) \) for each open set \( V \) of \( Y \).
2. If the composite \( g \circ f: X \to Z \) is weakly continuous and \( g: Y \to Z \) is an open bijection, then \( f \) is weakly continuous.

**Proof.** Statement (1) is Theorem 7 of [12]. We shall show Statement (2) by utilizing Statement (1). Let \( V \) be any open set of \( Y \). Since \( g \) is open, \( g(V) \) is open in \( Z \) and \( (g \circ f)^{-1}(g(V)) \subseteq (g \circ f)^{-1}(g(V)) \). Since \( g \) is bijective, \( (g \circ f)^{-1}(g(V)) = f^{-1}(V) \). Moreover, since \( g \) is open, \( (g \circ f)^{-1}(g(V)) = f^{-1}(V) \). Consequently, we obtain \( f^{-1}(V) \subseteq f^{-1}(V) \) and hence \( f \) is weakly continuous.

The following theorem is an improvement of [2, Theorem 3.4] and [11, Theorem 1].

**Theorem 3.3.** Let \((X, \tau)\) be a regular space with the fixed point property. If \( \sigma \) is a topology for \( X \) stronger than \( \tau \) and \( G(\tau) = G(\sigma) \) for each \( G \in \sigma \), then \((X, \sigma)\) has the wcFPP.

**Proof.** Let \( f: (X, \sigma) \to (X, \sigma) \) be any weakly continuous function. Let \( g: (X, \sigma) \to (X, \tau) \),
h : (X, ρ) → (X, ρ) and i : (X, ρ) → (X, σ) be the same functions as in Proof of Theorem 3.2. Since f = i ∘ g is weakly continuous and i is an open bijection, g is weakly continuous by Lemma 3.1. Since (X, ρ) is regular, g is continuous [8]. Next, we shall show that h is continuous. Let x ∈ X and V be an open set of (X, σ) containing h(x). Since (X, ρ) is regular, there exists G ∈ ρ such that h(x) ∈ G ⊆ G(σ) ∩ V. Since g is continuous, g⁻¹(G) ∈ σ and h⁻¹(G) = f⁻¹(G) = g⁻¹(G). Therefore, we have h⁻¹(G) = f⁻¹(G) ∈ σ. Since f is weakly continuous, by Lemma 3.1 f⁻¹(G) ⊆ f⁻¹(G(σ)). It follows from the same argument as in Proof of Theorem 3.2 that h is continuous. Since (X, ρ) has the fixed point property, there exists a point x ∈ X such that x = h(x) = f(x). This shows that f has the fixed point property.

COROLLARY 3.2 (Arya and Deb [2]). If (X, ρ) is a regular space with the fixed point property and if σ is a topology for X stronger than ρ such that σ ⊆ ρ for each G ∈ σ, then (X, σ) has the σ-continuous fixed point property.

ACKNOWLEDGEMENT: This paper was written during the second author stayed in Messina University for May and June 1988. He would like to thank to C.N.R and Messina University for its hospitality. The second author's research was supported by M.P.I. "fondi 40%" (ITALY).

REFERENCES