A NOTE ON THE k-DOMINATION NUMBER OF A GRAPH

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ABSTRACT. The k-domination number of a graph $G = (V,E)$, $\gamma_k(G)$, is the least cardinality of a set $X \subseteq V$ such that any vertex in $V \setminus X$ is adjacent to at least $k$ vertices of $X$.

Extending a result of Cockayne, Gamble and Shepherd [4], we prove that if $\delta(G) > \frac{n+1}{n} k - 1$, $n \geq 1$, $k \geq 1$ then, $\gamma_k(G) < \frac{np}{n+1}$, where $p$ is the order of $G$.

KEY WORDS AND PHRASES. k-dominating set and k-domination Number.

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1. INTRODUCTION.

A set $X$ of vertices of a graph $G = (V,E)$ is k-dominating if each vertex of $V \setminus X$ is adjacent to at least $k$ vertices of $X$. The k-domination number of a graph $G$, $\gamma_k(G)$, is the smallest cardinality of a k-dominating set of $G$.

We write $\delta = \delta(G)$ for the minimum degree of vertices in $G$ and $|G| = p$ is the number of vertices of $G$.

Several results concerning $\gamma_k(G)$ have been established by Fink and Jacobson [1], [2] who showed that $\gamma_k > \frac{kp}{\Delta + k}$, and recently by Favaron [3].

As for the upper bound, Cockayne, Gamble and Shepherd proved the following:

THEOREM 1.1. If $G$ has $p$ vertices and $\delta > k$, then $\gamma_k(G) < \frac{kp}{k+1}$.

2. MAIN RESULTS.

Our aim in this note is to extend Theorem 1.1 and give a shorter proof of that given in Cockayne, Gamble, and Shepherd [4]. We prove,
THEOREM 2.1. Let \( n, k \) be positive integers and \( G \) a graph such that

\[ \delta(G) > \frac{n+1}{n}k-1. \]

Then, \( \gamma_k(G) < \frac{np}{n+1} \).

PROOF. Let \( V_1, V_2, \ldots, V_{n+1} \) be a partition of \( V(G) \) into \( n+1 \) subsets which maximizes the number of edges in \( E' \) where \( E' = E(G) \setminus \bigcup_{i=1}^{n+1} E(<V_i>) \) and \( <V_i> \) is the subgraph induced on the vertex set \( V_i \).

By a classical argument of Erdős [5] we have that for every \( x \in V \), \( \deg_H(x) > \left[ \frac{n}{n+1} \deg_G(x) \right] \), where \( H = H(V',E') \), \( V' = V \), and \( E' \) is as above. Hence we conclude that:

\[ \deg_H(x) > \left[ \frac{n}{n+1} \left( \frac{n+1}{n}k-1 \right) \right] = \left[ k - \frac{n}{n+1} \right] = k. \]

Assume W.L.O.G. that \( |V_1| > |V_2| > \ldots > |V_{n+1}| \). Then the set \( A = \bigcup_{i=2}^{n+1} V_i \) is a \( k \)-dominating set of \( G \) since each vertex \( x \in V_i \) is adjacent to at least \( k \) vertices of \( A \). Thus it follows that \( \gamma_k(G) < p - |V_1| < \frac{np}{n+1} \).

COROLLARY 1. [4] If \( \delta(G) > k \) then \( \gamma_k(G) < \frac{kp}{k+1} \).

PROOF. Take \( n = k \) in Theorem 2.1.

COROLLARY 2: If \( \delta(G) > 2k - 1 \) then \( \gamma_k(G) < \frac{p}{2} \).

REMARK. Using a similar argument we can prove the following:

If \( \delta(G) > k > 1 \) and \( \chi(G) = n \), then \( \gamma_k(G) < \frac{(n-1)p}{n} \).

REFERENCES

2. FINK, J.F. and JACOBSON, M.S., On n-Domination, n-Dependence and Forbidden Subgroups, id. 301-311.