ABSTRACT. In continuing from previous papers, where we studied the existence and uniqueness of the global solution and its asymptotic behavior as time $t$ goes to infinity, we now search for a time-periodic weak solution $u(t)$ for the equation whose weak formulation in a Hilbert space $H$ is

$$\frac{d}{dt}(u',v) + \delta(u',v) + \alpha b(u,v) + \beta a(u,v) + (G(u),v) = (h,v)$$

where: $'= d/dt; (,)$ is the inner product in $H$; $b(u,v), a(u,v)$ are given forms on subspaces $U \subset W$, respectively, of $H$; $\delta > 0$, $\alpha > 0$, $\beta > 0$ are constants and $\alpha + \beta > 0$; $G$ is the Gateaux derivative of a convex functional $J: V \subset H \rightarrow [0, \infty)$ for $V = U$, when $\alpha > 0$ and $V = W$ when $\alpha = 0$, hence $\beta > 0$; $v$ is a test function in $V$; $h$ is a given function of $t$ with values in $H$.

Application is given to nonlinear initial-boundary value problems in a bounded domain of $\mathbb{R}^n$.

KEYWORDS AND PHRASES. Periodic weak solution, Gateaux derivative.

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1. INTRODUCTION.

In continuation of Brito [1], [2], where we studied existence and uniqueness of the global solution and its asymptotic behavior as time $t$ goes to infinity, we now search for a time-periodic weak solution $u(t)$, i.e., such that

$$u(0) = u(T); u'(0) = u'(T)$$

for the equation whose weak formulation in a Hilbert space $H$ is

$$\frac{d}{dt}(u',v) + \delta(u',v) + \alpha b(u,v) + \beta a(u,v) + (G(u),v) = (h,v)$$

where

$$= \frac{d}{dt}; (,) \text{ is the inner product in } H; b(u,v), a(u,v)$$

are given forms on subspaces $U \subset W$, respectively, of $H$; $\delta > 0$, $\alpha > 0$, $\beta > 0$ are constants and $\alpha + \beta > 0$; $G$ is the Gateaux derivative of a convex functional $J: V \subset H \rightarrow [0, \infty)$, for $V = U$, when $\alpha > 0$, and $V = W$, when $\alpha = 0$, hence $\beta > 0$; $v$ is a test function in $V$; $h$ is a given function of $t$ with values in $H$. 
Application is given to initial-boundary value problems in a bounded domain $\Omega$ of $\mathbb{R}^n$ for the following equations, in which $p > 2$ depends on $n$, $\alpha > 0$, $\beta > 0$, $k > 0$:

\[ u'' + \delta u' - \Delta u + |u|^{p-2}u = h \]  
\[ u'' + \delta u' + a\Delta^2 u - \beta \Delta u + u + |u|^{p-2}u = h \]  
\[ u'' + \delta u' + a\Delta^2 u - (\beta + k \int_{\Omega} (\nabla u)^2 d\Omega) \Delta u = h \]  

and the generalization of (1.4) in a Hilbert space $H$

\[ u'' + \delta u' + a\Delta^2 u + \beta \Delta u + M(|A|^{1/2})^2 \Delta u = h \]

for $A$ a linear operator in $H$, $M$ a real function.

For related problems, we refer to Birolli [3], Lovicar [4]; see also references in Brito [1].

2. PRELIMINARIES.

We consider three Hilbert spaces $U \subseteq W \subseteq H$ each continuously embedded and dense in the following.

We assume the injection $W \hookrightarrow H$ compact.

Let $(\cdot, \cdot)$ denote the inner product in $H$ and $\| \cdot \|$ its norm.

Let $a(u,v)$ and $b(u,v)$ be two continuous, symmetric, bilinear forms in $W$ and $U$, respectively. We shall write $a(v)$ for $a(v,v)$, $b(v)$ for $b(v,v)$. We shall assume that $(a(v))^{1/2}$ defines in $W$ a norm equivalent to the norm of $W$ and, similarly, that $(b(v))^{1/2}$ defines in $U$ a norm equivalent to the norm of $U$.

Let $c > 0$ be such that

\[ c|v|^2 < a(v) \text{ for } v \text{ in } W. \]  

Let $A$ be a linear operator in $H$, with domain $D(A)$ such that $U \subseteq D(A) \subseteq W$ and

\[ a(u,v) = (Au,v) \text{ for } u \text{ in } U, \ v \text{ in } W \]
\[ b(u,v) = (Au, Av) \text{ for } u, v \text{ in } U \]

$\delta > 0$, $\alpha > 0$, $\beta > 0$ are constants and $\alpha + \beta > 0$.

Assume

$V = U$ if $\alpha > 0$, $\beta > 0$;
$V = W$ if $\alpha = 0$, $\beta > 0$.

Consider a convex functional

$J: V \times [0, \infty) \text{ such that } J(0) = 0.$

Let $G: V \times H$ be the Gateaux derivative of $J$. We assume $G$ is Gateaux differentiable, locally Lipschitz and $G(0) = 0$.

With these hypothesis, we have, from [2] Theorem 3.1 and [1] Lemma 2.1, respectively
THEOREM 2.1. Given \( u_0 \) in \( V \), \( u_1 \) in \( H \), \( h \) in \( L^2(0,T;H) \), there is a unique function \( u \) such that

a) \( u \in L^\infty(0,T;V) \); \( u' \in L^\infty(0,T;H) \); \( G(u) \in L^\infty(0,T;H) \)

b) for all \( v \) in \( V \), \( u \) satisfies,
\[
\frac{d}{dt}(u',v) + \delta(u',v) + \alpha(u,v) + \beta(u,v) + (G(u),v) = (h,v) \tag{2.2}
\]

c) \( u \) satisfies the initial conditions
\[
u(0) = u_0; \quad u'(0) = u_1
\tag{2.3}
\]
d) \( u \) satisfies the energy equation
\[
E(t) + \int_0^t |u'(s)|^2 ds = E(0) + \int_0^t (h(s),u'(s)) ds \tag{2.4}
\]

where
\[
2E(t) = |u'(t)|^2 + \alpha(u(t)) + \beta(u(t)) + 2J(u(t)).
\]

THEOREM 2.2. In the conditions of Theorem 2.1, the map \( S: V \times H \rightarrow V \times H \) given by
\[
S(u_0,u_1) = (u(0),u'(0))
\]
is, for fixed \( t \), (sequentially) weakly continuous (i.e., if \( \phi_n \) \( v \) weakly in \( V \times H \), then \( S(\phi_n) \rightarrow S(\phi) \) weakly in \( V \times H \)).

We shall, further, assume that
\[
2J(v) - (G(v),v) < 0 \quad \text{for} \quad v \in V. \tag{2.5}
\]

3. EXISTENCE OF TIME-PERIODIC WEAK SOLUTIONS.

We shall refer to \( u(t) \) in the conditions of Theorem 1.1 as the solution of (2.2) with initial conditions \( (u_0,u_1) \) in \( V \times H \), given by (2.3).

THEOREM 3.1. If \( h \in C([0,T];H) \) there is at least one solution of (2.2) with initial condition in \( V \times H \) such that
\[
u(0) = u(T); \quad u'(0) = u'(T). \tag{3.1}
\]

PROOF. Take \( v = u(t) \) in (2.2) multiplied by constant \( 2\gamma > 0 \) and add it to the energy equation (2.4) differentiated and multiplied by 2, to obtain, with (2.5),
\[
\frac{d}{dt} \left( |u'|^2 + \alpha(u) + \beta(u) + 2J(u) + 2\gamma(u,u') \right) + \\
+ 2\gamma \left( |u'|^2 + \alpha(u) + \beta(u) + 2J(u) + 2\gamma(u,u') \right) + \\
+ 2(\delta-2\gamma) \left( |u'|^2 + \gamma(u,u') \right) < 2(h,u' + \gamma u).
\]

For \( 0 < \gamma < \delta/2 \), let
\[
w(t) = |u' + \gamma u|^2 + \alpha(u) + \beta(u) + 2J(u). \tag{3.2}
\]
Then we have
\[
w'(t) + 2\gamma w(t) < 2(h,u' + \gamma u) - 2(\delta-2\gamma)(u',u' + \gamma u) + \\
+ \gamma^2 \frac{d}{dt} |u|^2 + 2\gamma^3 |u|^2.
\]
The right-hand side of the above inequality is equal to
\[2(h,u' + yu) + 2(y^2u - (\delta - 2\gamma)u', u' + yu) =
2(h,u' + yu) - 2(\delta - 2\gamma)(u' + yu', u' + yu) + 2(\delta - \gamma)(yu, u' + yu).
\]

Therefore
\[w'(t) + 2yw(t) < 2(h,u' + yu) + \frac{(\delta - \gamma)^2 y^2 |u|^2}{2(\delta - 2\gamma)}.
\]

Observing (2.1) and (3.2), we obtain
\[w(t) > |u' + yu|^2 + \varepsilon |u|^2, \text{ with } \varepsilon = ac^2 + \beta c > 0.
\]

By assumption, \(\beta + \alpha > 0\).

We choose \(0 < \gamma < \delta/2\) so that
\[\rho = \frac{(\delta - \gamma)^2 y^2}{2(\delta - 2\gamma)} < 2\gamma.
\]

This is possible, because it amounts to choosing \(\gamma\) so that
\[B(\gamma) = (\delta - \gamma)^2 \gamma - 4 \varepsilon (\delta - 2\gamma) < 0.
\]

and \(\lim_{\gamma \to 0} B(\gamma) = -4 \varepsilon \delta < 0\).

It follows from (3.3), with (3.4), (3.5), that
\[w'(t) + 2yw(t) < 2|h(t)| \sqrt{w(t)} + \rho w(t).
\]

Hence for \(0 < t < T\),
\[w(t) < F(t)
\]
where
\[F(t) = e^{-2\gamma t} w(0) + \int_0^t e^{2\gamma s} [2|h(s)| \sqrt{w(s)} + \rho w(s)] ds.
\]

Therefore, because of (3.6), we have
\[F'(t) < (\rho - 2\gamma)|F(t) + 2|h(t)| F(t).
\]

Let \(F(t) = r\), with
\[r > \max_{0 < t < T} \frac{2|h(t)|}{2\gamma - \rho}.
\]

Then \(F'(t) < 0\). It follows, using (3.6), (3.7), that if \(w(0) < r\) then \(w(t) < r\),
for \(0 < t < T\).

Consider
\[K = \{(u_0, u_1) \in V \times H; |u_1 + yu_0|^2 + ab(u_0) + \beta a(u_0) + 2J(u_0) < \tau\}.
\]

We proved that the map \(S: V \times H \times V \times H\) given by
\[S(u_0, u_1) = (u(T), u'(T))
\]
takes \(K\) into \(K\).

It is easy to check that \(K\) is a nonempty, closed, bounded, convex subset of \(V \times H\).

The fact that \(S\) has a fixed point, i.e., that (3.1) holds for some \((u_0, u_1) \in K\),
now follows from Theorem 2.2 as a consequence of the well-known fixed point Theorem:
Let $B$ be a separable, reflexive Banach space, $K$ a nonempty closed, bounded convex subset of $B$, and $S$ a (sequentially) weakly continuous operator of $K$ into $K$. Then $S$ has at least one fixed point in $K$.

4. APPLICATIONS.

We devote this Section to applications of Theorem 3.1 involving initial-boundary value problems in a bounded domain $\Omega$ with regular boundary in $\mathbb{R}^n$ for equations (1.2), (1.3), (1.4).

In what follows

$H = L^2(\Omega)$, $W = H^1_0(\Omega)$, and

$a(u,v) = (\nabla u, \nabla v) = \int_{\Omega} \nabla u \nabla v \, d\Omega.$

Let

$A = -\Delta$, $D(A) = H^1_0(\Omega) \cap H^2(\Omega)$,

$b(u,v) = (\Delta u, \Delta v)$, $U = H^1_0(\Omega) \cap H^2(\Omega)$.

Note that similar results are obtained if we suppose $U = H^2_0(\Omega)$.

For $\delta > 0$, $h \in C([0,T]; H)$, we have

EXAMPLE 4.1. Let $\alpha = 0$, $\beta = 1$, $V = W = H^1_0(\Omega)$ and

$J(u) = \frac{1}{p} |u|^p L^p(\Omega)$ for $u \in V$

where $2 < p < 2(n-1)/(n-2)$ if $n > 2$; $p > 2$ if $n < 2$. Then $J(u)$ is well-defined in $V$ and

$G(u) = |u|^{p-2} u \in H.$

We refer to [2], example 5.1, for the proof. It is clear that (2.5) holds. Therefore Theorem 3.1 ensures existence of a $T$-periodic weak solution of

$u'' + u' - \Delta u + |u|^{p-2} u = h.$

EXAMPLE 4.2. For $\alpha > 0$, $\beta > 0$, $V = U = H^1_0(\Omega) \cap H^2(\Omega)$ (or $H^2_0(\Omega)$), let

$J(u) = \frac{1}{p} |u|^p_{L^p(\Omega)} + \frac{1}{2} |u|^{2}_{L^2(\Omega)}$ for $u \in V$

where

$2 < p < 2(n-2)/(n-4)$ if $n > 4$; $p > 2$ if $n < 4$.

Then $J(u)$ is well-defined in $V$ and

$G(u) = |u|^{p-2} u + u \in H.$

We refer to [2], example 5.2, for the proof. It is clear that (2.5) holds. Then Theorem 3.1 ensures existence of a $T$-periodic weak solution of

$u'' + \delta u' + a \Delta u - \beta u + |u|^{p-2} u = h.$
EXAMPLE 4.3. For $\theta > 0$, $\beta > 0$, $k > 0$, $V = U = H^1_0(\Omega) \cap H^2(\cdot)$ (or $H^2_0(\Omega)$), let

$$J(u) = \frac{k}{4} (a(u))^2$$

for $u$ in $V$. Then

$$G(u) = -ka(u)u \in H.$$ 

We refer to [2], Example 5.3, for the proof. It is clear that (2.5) holds.

Thus Theorem 3.1 ensures existence of a $T$-periodic weak solution of

$$u'' + \delta u' + \alpha^2 u = \{\beta + k \int (\nabla u)^2 d\Omega\} u = h.$$ 

Generalizing, let $M$ be a $C^1$ function such that, for $s > 0$,

$$M(s) > k_s$$

and $M'(s) > 0$.

Take $V = U$ and $A$ as in Section 1. Let

$$J(u) = \frac{1}{2} \int_0^T M(s)ds$$

for $u$ in $V$. Then

$$G(u) = M(a(u)), Au \in H.$$ 

We refer to [2], Example 5.3, for the proof. It is clear that (2.5) holds. Therefore Theorem 3.1 ensures existence of a $T$-periodic weak solution of

$$u'' + \delta u' + \alpha^2 u = \{\beta + M(A^{1/2}u^2)\} Au = h.$$ 

REFERENCES