ON NORMAL LATTICES AND WALLMAN SPACES

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ABSTRACT. Let X be an abstract set and \( \mathcal{M} \) a lattice of subsets of X. The notion of \( \mathcal{M} \) being mildly normal or slightly normal is investigated. Also, the general Wallman space with an alternate topology is investigated, and for \( \mathcal{M} \) not necessarily disjunctive, an analogue of the Wallman space is constructed.

KEY WORDS AND PHRASES. Normal lattice, 0-1 valued measures, Wallman space, disjunctive lattice, almost compact, almost countably compact.

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1. INTRODUCTION.

In the first part of this paper, we consider lattices which satisfy conditions weaker than normality; more precisely mildly normal and slightly normal lattices. We give examples of such lattices, and then investigate the preservation of these properties under lattice extension and restriction.

Next, we investigate spaces which are related to the general Wallman space. First, instead of considering the customary topology on the Wallman space, we introduce another topology and show how topological properties reflect strongly to the underlying lattice. Then we consider the case of a lattice which is not necessarily disjunctive and construct an associated Wallman type space. This work generalizes that of Liu (see Section 5).

We adhere to standard lattice terminology that can be found, for example, in [1], [2], [3], [4], [5]. However, in section 2, we summarize the principal lattice concepts and notations that will be used throughout the paper for the convenience of the reader. We then precede to the consideration of mildly normal and slightly normal lattices in section 3, and then to analogues of the general Wallman space in sections 4 and 5. We finally note that most of the results hold equally well for abstract lattices.

2. DEFINITIONS AND NOTATIONS.

a) Let X be an abstract set and \( \mathcal{M} \) a lattice of subsets of X. We shall always assume, without loss of generality for our purposes, that \( \phi, X \in \mathcal{M} \). The set whose general element \( L' \) is the complement of an element \( L \) of \( \mathcal{M} \) is denoted by \( \neg L \).
said to be complement generated iff, for every $L$ of $\Omega$ there exists a sequence $(L_n)_{n=1}^\infty$ in $\Omega$ such that $L = \bigcap_{n=1}^\infty L_n$. $\Omega$ is separating iff, for any two elements $x \neq y$ of $X$, there exists an element $L \in \Omega$ such that $x \notin L$ and $y \notin L$. $\Omega$ is regular iff, for any two elements $x \neq y$ of $X$, there exists $A, B \in \Omega$ such that $x \in A$, $y \in B$ and $A \cap B = \phi$. $\Omega$ is said to be disjunctive iff, for every $x \in X$ and every $L \in \Omega$, if $x \notin L$ then there exists $L_1 \in \Omega$ such that $x \in L_1$ and $L \subseteq L_1$. $\Omega$ is normal iff, for any $L_1, L_2 \in \Omega$, if $L_1 \cap L_2 = \phi$ then there exists $L_1, L_2 \in \Omega$ such that $L_1 \subseteq L_1'$, $L_2 \subseteq L_2'$ and $L_1 \cap L_2' = \phi$. $\Omega$ is Lindelöf iff, for every $L \in \Omega$, $\alpha \in \Omega$, if $\bigcap_{\alpha} L = \phi$ then for any finite subcollection $(L_\alpha)_{\alpha=1}^n$ of $(L_\alpha)_{\alpha=1}^\infty$, $\bigcap_{\alpha=1}^n L_\alpha = \phi$. $\Omega$ is compact iff, for every $L \subseteq \Omega$, if $L$ is compact then there exists $L_1, L_2 \in \Omega$ such that $L_1 \subseteq L$, $L_2 \subseteq L$ and $L = L_1 \cap L_2'$. Next, consider any two lattices $\Omega_1, \Omega_2$ of subsets of $X$. $\Omega_1$ is said to semi-separate $\Omega_2$ or for abbreviation $(\Omega_1, s.s. \Omega_2)$ iff, for every $L_1 \in \Omega_1$ and every $L_2 \in \Omega_2$ if $L_1 \cap L_2 = \phi$ then there exists $L_1, L_2 \in \Omega_1$ such that $L_1 \subseteq L_1'$ and $L_2 \subseteq L_2'$. $\Omega_1$ is said to separate $\Omega_2$ iff, for any $L_1, L_2 \in \Omega_1$, if $L_2 \cap L_2' = \phi$ then there exists $L_1, L_1 \in \Omega_1$ such that $L_2 \subseteq L_1$, $L_2 \subseteq L_1'$ and $L_1 \cap L_1' = \phi$. We denote by $\Omega^{\Omega_1, s.s. \Omega_2}$ a set whose general element is the intersection of arbitrary subsets of $\Omega$. b) Let $\Lambda \subseteq \Omega$ be any algebra of subsets of $X$. A measure on $\Lambda$ is defined to be a function, $\mu$, from $\Lambda$ to $R$ such that $\mu$ is bounded and finitely additive. The algebra of subsets of $X$ generated by $\Omega$ is denoted by $\Lambda(\Omega)$. If $x \in X$, then $\mu_x$ is the measure concentrated at $x$, so $\mu_x(A) = \frac{1}{n}$ if $x \in A$, where $A \subseteq \Omega$. The set whose general element is a measure on $\Lambda(\Omega)$ is denoted $M(\Omega)$. Note that, since every element of $M(\Omega)$ is equal to the difference of nonnegative elements of $M(\Omega)$, without loss of generality, we may work exclusively with nonnegative elements of $M(\Omega)$. Let $\mu \in M(\Omega)$, $\mu$ is $\Omega$-regular if for any $A \in \Omega$; $\mu(A) = \sup\{\mu(L) | L \subseteq A, L \in \Omega\}$. The set whose general element is a measure on $\Omega$ is denoted $M(\Omega)$. Note that, since every element of $M(\Omega)$ is equal to the difference of nonnegative elements of $M(\Omega)$, without loss of generality, we may work exclusively with nonnegative elements of $M(\Omega)$. Let $\mu \in M(\Omega)$, $\mu$ is $\Omega$-regular if for any $A \in \Omega$; $\mu(A) = \sup\{\mu(L) | L \subseteq A, L \in \Omega\}$.
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I \( u \in R \) is regular iff, for any \( i, j \in I(R) \), \( I(B_i) \subseteq I(B_j) \) then \( S(I) \subseteq S(J) \). The result \( u \in I(R) \) iff \( I \) is disjunctive leads us to the Wallman Topology which is obtained by taking the totality of all \( W(L) = \{ u \in I(R) \mid u(L) = 1 \} \) as a base for the closed sets on \( I(R) \). For a disjunctive \( I \), \( I(R) \) with the \( T \) of closed sets is a compact \( T_1 \) space and will be \( T_2 \) iff \( I \) is normal and is called the general Wallman space associated with \( X \) and \( I \). Also, for a disjunctive \( I \) and \( A, B \in A(I) \), \( W(A) \) is a lattice with respect to union and intersection. Moreover, \( W(A') = (W(A))' \), \( W(A(I)) = (W(I)) \), \( W(A) = W(B) \) iff \( A = B \) and \( W(A) \subseteq W(B) \) iff \( A \subseteq B \). Now, we note that, if \( I \) is disjunctive so is \( W(I) \), and in addition to each \( \mu \in M(I) \), there exists a \( \mu(A) \in M(W(I)) \) defined by \( \mu(A) = \mu(W(I)) \) for all \( A \in A(I) \) such that the map \( \mu + \mu \) is one-to-one and onto; moreover \( \mu \in M(R) \) iff \( \mu \in M(W(I)) \).

3. ON NORMAL LATTICES.

In this section, we elaborate on the notion of a normal lattice, investigate lattices which satisfy weaker conditions, and discuss their interrelations under the extension and restriction properties. Throughout this section \( I, I_1, I_2, I_3 \) will denote lattices of subsets of the set \( X \). Note that, for \( L_1 \subseteq I \), \( i = 1, 2, 3 \); \( I \) is normal iff \( L_1 \subseteq L_2 \), then \( L_1 = A \cup B \), \( A \subseteq L_1 \); \( B \subseteq L_2 \); \( A, B \in I \).

THEOREM 3.1. Let \( I \) be normal, \( \mu \in I(I) \) and \( G \subseteq \{ L' \in I(I) \mid \mu(L') = 1, L' \in I \} \) then, \( G \) is a prime \( I \)-filter.

PROOF. Clearly, \( G \) is a \( I \)-filter since \( \mu \not\in G \), \( \mu(L_1 \cap L_2) = 1 \) and if \( L_1 \subseteq G \), \( L_1 \subseteq L_2 \) then \( L_1 \in G \). In addition, if \( L_1 \cup L_2 \subseteq G \), then \( L_1 \subseteq L_1 \cup L_2 \) and by the normality of \( I \), \( L_1 \cup L_2 \subseteq L \) with \( L_1 \subseteq L_1 \cap L_2 \) then either \( \mu(L_1) = 1 \) or \( \mu(L_2) = 1 \) and so either \( L_1 \subseteq G \) or \( L_2 \subseteq G \). Thus, \( G \) is a prime \( I \)-filter.

THEOREM 3.2. Suppose \( I_1 \subseteq I_2 \) and \( I_1 \) separates \( I_2 \). Then, \( I_1 \) is normal iff \( I_2 \) is normal.

PROOF. (i) Suppose \( I_1 \) is normal then by the separation, it is clear that \( I_2 \) is normal. (ii) Suppose \( I_2 \) is normal. Let \( \mu \in I(I_1) \), \( \mu < \nu_1, \nu_2(I_2) \) then \( \nu_1, \nu_2 \in I(I_1) \). Extend \( \nu_1, \nu_2 \) to \( \lambda_1, \lambda_2 \subseteq I(I_2) \) and \( \mu \) to \( \tau \in I(I_2) \). Suppose there exists an \( L_2 \subseteq I_2 \); \( \tau(L_2) = 1 \), but \( \lambda_1(L_2) = 0 \), then \( \lambda_1(L_2) = 1 \) and so there exists \( \mu(L_2) = \mu(L_2) = 1 \). By the separation, there exists \( \hat{L}_1 \subseteq I_1 \), \( \mu(L_1) \neq \mu(L_1) \), \( \tau(L_1) = 1 \) but \( \tau(L_1) = 0 \). By the normality of \( I_2 \), \( \lambda_1(L_1) = 1 \) which is a contradiction. Therefore, \( \tau \not\in I_1 \) and similarly \( \tau \not\in I_2 \). Thus, by the normality of \( I_2, \lambda_1 = \lambda_2 \) then \( \nu_1 = \nu_2 \) and so \( I_1 \) is normal.

DEFINITION 3.1. \( I \) is said to be mildly normal, if for all \( \mu \in I(I) \), there exists a unique \( \nu \in I(I) \) such that \( \mu < \nu(A) \).

DEFINITION 3.2. \( I \) is said to be almost countably compact, if for all \( \mu \in I(I) \), \( \mu \in I(I) \).

DEFINITION 3.3. \( I \) is said to be prime replete, if for all \( \mu \in I(I) \), \( S(\mu) \neq \phi \).

THEOREM 3.3. If \( I \) is regular and prime replete, then \( I \) is mildly normal.

PROOF. If \( I \) is not mildly normal, then there exists \( \mu \in I(I) \), \( \nu_1 \neq \nu_2 \in I(I) \) and \( \mu \not\in I_1 \), \( \mu \not\in I_2 \). Then, there exists \( L_1, L_2 \subseteq I \); \( L_1 \cap L_2 = \phi \), \( \nu_1(L_1) = \nu_2(L_2) = 1 \) and \( \nu_1(L_2) = \nu_2(L_1) = 0 \). But since \( I \) is regular, \( S(\mu) = S(\nu_1) \cup L_1 \) and \( S(\mu) = S(\nu_2) \cap L_2 \). Then, \( S(\mu) \subseteq (L_1 \cap L_2) = \phi \) and so \( S(\mu) = \phi \). Since \( I \) is prime replete, \( \mu \) must be mildly normal.
THEOREM 3.4. If \( \alpha \) is regular and Lindelöf, then \( \Omega \) is mildly normal.

PROOF. If \( \alpha \) is not mildly normal, then by the proof of Theorem 3.3, \( S(\mu \cap L) = \emptyset \). Moreover, \( \phi = \{ L \cap \Omega \cap \Omega \} = S(\mu_1) = S(\mu_2) \) with \( \mu(L) = 1 \) and \( \mu(\alpha) = 1 \). Since \( \Omega \) is Lindelöf \( \cap \Omega \) and \( \cap \Omega = \emptyset \), (may assume \( L_1 \)), but \( \mu(\alpha) = 1 \) for all \( \alpha \) which is a contradiction since \( \mu \in I_{\alpha}(\Omega) \).

Thus, \( \Omega \) must be mildly normal.

THEOREM 3.5. If \( \Omega \) is almost countably compact and mildly normal, then \( \Omega \) is normal.

PROOF. Suppose \( \Omega \) is almost countably compact, and \( \mu \in I(\Omega) \). Then \( \mu \in I(\Omega') \) on \( \Omega' \), and so \( \mu \in \mu(\Omega) \); \( \mu \in \mu(\Omega) \). If \( \phi \in I(\Omega) \) and \( \mu \in \phi(\Omega) \), then \( \mu \in \phi(\Omega) \).

THEOREM 3.6. Suppose \( \Omega_1 \cap \Omega_2 \) and \( \Omega_2 \) separates \( \Omega_2 \). Then, \( \Omega_2 \) is mildly normal if \( \Omega_1 \) is mildly normal.

PROOF. Suppose \( \mu \in I(\Omega_1) \), \( \mu \in \nu(\Omega_2) \), \( \mu \in \nu(\Omega_2) \), \( \nu_1 \cap \Omega_2 \) and \( \nu_2 \cap \Omega_2 \). Since \( \Omega_1 \) is mildly normal the rest of the proof is obvious.

DEFINITION 3.4. \( \Omega_2 \) is said to be \( \Omega_1 \)-countably bounded, if given \( B, B \in \Omega_2 \), there exists an \( A \in \Omega_1 \); \( A \in \phi \) and \( B \in A \).

In the next theorem, we will see when a partial converse of the theorem 3.6 is true.

THEOREM 3.7. Suppose \( \Omega_1 \) separates \( \Omega_2 \), and \( \Omega_2 \) is \( \Omega_1 \)-countably bounded, then if \( \Omega_2 \) is mildly normal so is \( \Omega_1 \).

PROOF. Suppose \( \Omega_2 \) is mildly normal and \( \Omega_1 \) separates \( \Omega_2 \). Let \( \mu \in I(\Omega_1) \) and \( \nu_1 \cap \Omega_1 \), \( \nu_2 \cap \Omega_2 \) where \( \nu_1 \), \( \nu_2 \in I(\Omega_2) \). Extend \( \nu_1 \) to \( \nu_1 \) and \( \nu_2 \) to \( \nu_2 \) respectively, where \( \nu_1 \in I(\Omega_2) \), and \( \nu_2 \in I(\Omega_2) \). Since \( \Omega_2 \) is \( \Omega_1 \)-countably bounded, there exists \( \Omega_2 \cap \Omega_2 \in \Omega_1 \). Then \( \nu_2 \) is \( \nu(\Omega_1) \) and \( \nu_2 \) is unique.

DEFINITION 3.5. \( \Omega_2 \) is said to be countably paracompact if, whenever \( \phi \), \( \phi \in \phi \), there exists \( B \in \Omega_1 \) and \( B \in \phi \).

DEFINITION 3.6. \( \Omega \) is said to be slightly normal, if for all \( \mu \in I(\Omega) \), there exists a unique \( \nu \in I(\Omega) \); \( \nu \in \nu(\Omega) \).

THEOREM 3.8. If \( \alpha \) is regular and Lindelöf, then \( \alpha \) is slightly normal.

PROOF. Suppose \( \alpha \) is not slightly normal. Let \( \mu \in I(\Omega) \), \( \mu \in \nu(\Omega) \), \( \mu \in \nu(\Omega) \) where \( \nu_1 \), \( \nu_2 \in I(\Omega) \); \( \nu_1 \), \( \nu_2 \in I(\Omega) \). Then \( \nu_1 \) is a measure, i.e. \( \nu \in \mu(\Omega) \). Moreover, \( \nu(\Omega) = \nu(\Omega) \) since \( \alpha \) is regular.

Also \( \nu \in \nu(\Omega) \), and since \( \alpha \) is Lindelöf, \( \nu(\Omega) \neq \emptyset \) which is a contradiction. Thus, \( \alpha \) must be slightly normal.

THEOREM 3.9. Suppose \( \nu_1 \) separates \( \nu_2 \). If \( \nu_1 \) is slightly normal, then \( \nu_2 \) is slightly normal.

PROOF. Let \( \mu \in I(\Omega) \); \( \mu \in \nu_1(\Omega) \), \( \mu \in \nu_2(\Omega) \) where \( \nu_1 \), \( \nu_2 \in I(\Omega) \), then by separation \( \nu_1 \cap \Omega_2 \), \( \nu_2 \cap \Omega_2 \), \( \mu \in I(\Omega) \) and \( \nu_1 \), \( \nu_2 \in I(\Omega) \). Therefore,
\( v_1 = v_2 \) since \( \Omega \) is slightly normal and so \( v_1 = v_2 \) by separation and hence \( \Omega_2 \) is slightly normal.

**Lemma 3.1.** If \( \Omega \) is complement generated, then \( \Omega \) is slightly normal.

**Proof.** If \( \Omega \) is complement generated, then \( I_\Omega(\Omega') \subset I_R(\Omega) \), that is, if \( \mu \in I_\Omega(\Omega') \) there exists a unique \( \nu \in I_R(\Omega); \nu < \chi(\Omega) \). Thus, \( \Omega \) is slightly normal.

In the next theorem, we will see when a partial converse of theorem 3.9 is true.

**Theorem 3.10.** Suppose \( \Omega \) separates \( \Omega \). If \( \Omega \) is mildly normal and countably paracompact then \( \Omega \) is slightly normal.

**Proof.** Let \( E \in \Omega \), \( \Omega \subset \Omega \) then \( E \subset \Omega \) since \( \Omega \) is countably paracompact and \( \Omega \) separates \( \Omega \). Therefore \( \Omega \) is countably paracompact and \( \Omega \) is \( \Omega \)-countably paracompact and \( \zeta(\Omega) < \zeta(\Omega) \). Since \( \mu \in I_\Omega(\Omega') \), then \( \zeta \in I_\Omega(\Omega') \). Now extend \( \nu_1, \nu_2 \) to \( \tau_1, \tau_2 \in I_R(\Omega) \) then \( \zeta < \tau_1(\Omega) \), \( \zeta < \tau_2(\Omega) \) by separation and so \( \tau_1 = \tau_2 \) since \( \Omega \) is mildly normal. Thus \( \nu_1 = \nu_2 \) and so \( \Omega_1 \) is slightly normal.

**Remark 3.1.** It is not difficult to give similar conditions (as in theorem 3.10) to obtain the other partial converse of theorem 3.9.

4. ON SPACES RELATED TO THE GENERAL WALLMAN SPACE.

We recall from section 2 that for an arbitrary lattice \( \Omega \) of subsets of \( X \), \( I_R(\Omega) \) with the topology \( \mathcal{W}(\Omega) \) of the closed set, is a compact \( T_2 \) space. Also, if \( \Omega \) is disjunctive and separating then \( X \) can be embedded in \( I_R(\Omega) \). Moreover, if \( \Omega \) is disjunctive, so is \( \mathcal{W}(\Omega) \), and \( I_R(\Omega) \) with the topology \( \mathcal{W}(\Omega) \) is \( T_2 \) iff \( \Omega \) is normal. In this section, we consider alternate topologies on \( I_R(\Omega) \).

**Theorem 4.1.** Consider \( \delta = \tau \mathcal{W}(\Omega') \) for a base of closed sets \( W(L'); L' \in \Omega \). Then \( I_R(\Omega) \) with the topology \( \delta \) is \( T_2 \).

**Proof.** If \( \mu_1 \neq \mu_2 \), then \( \mu_1, \mu_2 \in I_R(\Omega) \) then there exists \( L_1, L_2 \in \Omega; L_1 \cap L_2 = \phi \). Therefore \( \mu_1, \mu_2 \) are disjoint and \( \mu \in W(L_1) \) and \( \mu \notin W(L_2) \). Thus, \( I_R(\Omega) \) is \( T_2 \).

**Definition 4.1.** \( \Omega \) is said to be almost compact, if for all \( \mu \in I_R(\Omega') \), \( S(\mu) \neq \phi \).

**Theorem 4.2.** The lattice \( \mathcal{W}(\Omega') \) is almost compact.

**Proof.** Let \( \lambda \in I_R(\mathcal{W}(\Omega')) \) then \( \lambda = \mu \in I_R(\Omega) \) (section 2) and so \( S(\mu) = \mathcal{W}(L) \). Thus, \( \mu \in W(L), \mu \in S(\mu) \) on \( \mathcal{W}(\Omega) \) and so \( I_R(\Omega) \) with \( \mathcal{W}(\Omega) \) is almost compact.

**Remark 4.1.** We note (i) \( S(\lambda) = S(\mu) = \{\mu\} \). (ii) If \( \Omega \) is disjunctive, it is then clear that for any \( L \in \Omega \), \( L = \cap L', L \subset L' \in L \), \( L \in \Omega \).

**Theorem 4.3.** The sets of \( \mathcal{W}(\Omega) \) are clopen in the \( \delta \) topology.

**Proof.** Since \( W(L) \) is disjoint then by remark 4.2 (ii) for any \( L \in \Omega \), \( W(L) = \cap W(L'), W(L) \subset W(L'), L \in \Omega \) is closed in the \( \delta \) topology. But, \( W(L) \) is also open in \( \delta \) topology since \( W(L) = (W(L'))' \). Thus, the sets of \( \mathcal{W}(\Omega) \) are closed in the \( \delta \) topology.

**Theorem 4.4.** \( I_R(\Omega) \) with the \( \delta \) topology is compact iff \( \Omega \) is an algebra.

**Proof.** (i) Suppose \( I_R(\Omega) \) with \( \delta \) is compact. Thus, \( \mathcal{W}(\Omega') \) is a compact lattice and so is \( \mathcal{W}(\Omega) \). Let \( \xi \in I(W(\Omega')). \) Then \( S(\xi) = \phi \) on \( \mathcal{W}(\Omega), \xi = \phi, \lambda \in I(\Omega). \)
Let $\mu \in S(\zeta)$, $\mu \in I_R(\Omega)$ then $1 = \zeta(W(L'))$ iff $\zeta(W(L')) = 1$ iff $\lambda(L') = 1$.

Now $\lambda(L') = 1$. Then $\zeta(W(L')) = 1$ then $\mu(L') = 1$ then $\mu(L') = 1$. Thus, $\lambda < \mu(\Omega')$ and hence $\mu < \lambda(L)$. But $\mu \in I_R(\lambda)$ then $\mu = \lambda$. Thus $\mu = \lambda$ and hence $I(W(\Omega)) = I_R(W(\Omega))$. Thus, $\Omega = \Omega'$ [2]. (ii) The converse is clear.

THEOREM 4.4. $\delta = \tau(W(\Omega))$ iff $\Omega$ is an algebra.

PROOF. (i) Since the sets $W(\Omega)$ are clopen by Theorem 4.3 then $W(\Omega) \in \delta$ and so $\tau(W(\Omega)) \in \delta$. Now, if $\delta = \tau(W(\Omega))$ then since $\tau(W(\Omega))$ is compact so is $\delta$ and so $\Omega$ is an algebra by theorem 4.4. (ii) The converse is clear.

In the next theorem, we give another equivalent condition for $\Omega$ to be an algebra.

THEOREM 4.6. $\Omega$ is an algebra iff $W(\Omega')$ is a disjunctive lattice in $I_R(\Omega)$.

PROOF. (i) Suppose $\Omega$ is an algebra, then $I_R(\Omega) = I(\Omega)$. Let $\mu \in I_R(\Omega)$; $\mu \in W(L')$, $L \in \Omega$. Then $\mu(L') = 0$, $\mu(L) = \mu((L')')$. $L' \in \Omega$ and $W(L') \cap W((L')') = \emptyset$. Thus, $W(\Omega')$ is disjunctive. (ii) Suppose $W(\Omega')$ is disjunctive. Let $\mu \in I(\Omega)$ then there exists a $v \in I_R(\Omega)$; $\mu < \tau(v)$. For $\mu \neq v$, there exists $L \in \Omega$, $\mu(L) = 0$, $\forall(L) = 1$ then $v \notin W(L')$. Hence, by disjunctiveness $v \notin W(L')$, $L \in \Omega$ and $\phi = W(L') \cap W(L') = W(L' \cap L') = W(L' \cup L')$. Hence, $L \cup L' = X$, $L \cap L' = \emptyset$ and so $L' \subseteq L$, but since $\mu(L) = 1$ and $v \in W(L')$, $\forall(L) = 0$ which is a contradiction, since $\mu \neq v$. Thus, $I_R(\Omega) = I(\Omega)$, and so $\Omega$ is an algebra.

5. ON NON-DISJUNCTIVE LATTICES.

We next consider the case where $\Omega$ is not necessarily disjunctive. We begin, by introducing the notion of an $\Omega$-convergent measure and some related results and then proceed to the construction of an analogue of the Wallman space.

DEFINITION 5.1. $\mu \in I(\Omega)$ is said to be $\Omega$-convergent if there exists an $x \in X$ such that $\mu_x \leq \mu(\Omega)$.

THEOREM 5.1. $\mu$ is $\Omega$-convergent iff $S(\mu) \neq \emptyset$ on $n'$, for all $\mu \in I(\Omega)$.

PROOF. (i) Suppose $\mu$ is $\Omega$-convergent. Then there exists $x \in X$; $\mu_x < \mu(\Omega)$ and so $\mu < \mu_x(\Omega')$. Moreover $x \in S(\mu_x) \cap S(\mu) on \Omega'$. Thus, $S(\mu) \neq \emptyset$ on $\Omega'$. (ii) Suppose $S(\mu) \neq \emptyset$ on $\Omega'$ for $\mu \in I(\Omega)$. Let $x \in S(\mu)$ on $\Omega'$. Then, $\mu < \mu_x(\Omega')$ and so $\mu_x < \mu(\Omega)$.

THEOREM 5.2. Suppose $\mu_1 < \mu_2(\Omega)$, for all $\mu_1, \mu_2 \in I(\Omega)$. Then

a) If $\mu_1$ is $\Omega$-convergent so is $\mu_2$.

b) If $\Omega'$ is regular and $\mu_2$ is $\Omega$-convergent then $\mu_1$ is $\Omega$-convergent.

PROOF. a) Suppose $\mu_1$ is $\Omega$-convergent, then $\mu_x \leq \mu_1(\Omega)$ for some $x$, but $\mu_1 \leq \mu_2(\Omega)$ then $\mu_x \leq \mu_2(\Omega)$ and so $\mu_2$ is $\Omega$-convergent. b) Suppose $\mu_2$ is $\Omega$-convergent, then $\mu_x \leq \mu_2(\Omega)$ for some $x$ and so $\mu_2 < \mu_2(\Omega')$ and $x \in S(\mu_2)$ on $\Omega'$ but $\mu_1 \leq \mu_2(\Omega)$ then $\mu_2 < \mu_1(\Omega')$ and since $\Omega'$ is regular $x \in S(\mu_2) = S(\mu_1)$ on $\Omega'$. Thus, $S(\mu_1) \neq \emptyset$ on $\Omega'$ and so $\mu_1$ is $\Omega$-convergent.

THEOREM 5.3. Suppose $\Omega'$ is $T_2$ and $\mu$ is $\Omega$-convergent where $\mu \in I(\Omega)$. Then, there exists a unique $x \in X; \mu_x < \mu(\Omega')$.

PROOF. If $\Omega'$ is $T_2$ and if $x \neq y$ then there exists $L_1, L_2 \in \Omega; x \in L_1, y \in L_2$; $L_1 \cap L_2 = \emptyset$ and $\mu_x(L_1) = 1, \mu_y(L_2) = 1$. Now, if $\mu_x < \mu(\Omega)$ and $\mu_y < \mu(\Omega)$ then $\mu(L_1) = \mu(L_2) = 1$ but $L_1 \cap L_2 = \emptyset$ which is a contradiction and so the desired result is true.
DEFINITION 5.2. A is said to be weakly compact if for all \( \mu \in \mathcal{I}_R(\Omega) \), \( \mu \) is \( \Omega \)-convergent.

DEFINITION 5.3. A is said to be almost compact if for any \( \mu \in \mathcal{I}_R(\Omega') \), \( S(\mu) \neq \emptyset \) on \( \Omega \).

THEOREM 5.4. A is weakly compact iff \( \Omega' \) is almost compact.

PROOF. (i) Let \( \mu \in \mathcal{I}_R(\Omega) \) then since \( \Omega \) is weakly compact, there exists an \( \lambda \in X \) such that \( \mu \lambda < \mu(\Omega) \) and \( S(\mu) \neq \emptyset \) on \( \Omega' \). Thus, \( \Omega' \) is almost compact. (ii) Let \( \mu \in \mathcal{I}_R(\Omega) \) then \( S(\mu) \neq \emptyset \) on \( \Omega' \) since \( \Omega \) is almost compact. Let \( \lambda \in S(\mu) \) on \( \Omega' \) then \( \mu \in S(\mu) \), \( \mu \lambda < \mu(\Omega) \) and so \( \Omega \) is weakly compact.

Now, note that a topological space \( X \) is absolutely closed (generalized absolutely closed) iff the lattice \( \Theta \) of open sets is \( T_2(T_0) \) and weakly compact. Next, let \( \Omega \) be a lattice of subsets of \( X \) and define \( U(\Omega) = \{ \bigcup \alpha \in \Omega \} \).

THEOREM 5.5. a) If \( \Omega_1 \subseteq \Omega_2 \) and \( \Omega_2 \) is weakly compact, then \( \Omega_1 \) is weakly compact. b) Suppose \( \Omega_1 \subseteq \Omega_2 \subseteq U(\Omega_1) \) and \( \Omega_2 \) semi-separates \( \Omega_1 \). Then, if \( \Omega_1 \) is weakly compact, \( \Omega_2 \) is weakly compact.

PROOF. a) Extend \( \mathcal{I}_R(\Omega) \) to \( \mathcal{I}_R(\Omega) \). Since \( \Omega_2 \) is weakly compact, there exists an \( \lambda \in X \) such that \( \mu \lambda < \mu(\Omega_1) \) and so \( \Omega_1 \) is weakly compact. b) Let \( \nu \in \mathcal{I}_R(\Omega_2) \). Then since \( \Omega_1 \subseteq \Omega_2 \), \( \nu \in \mathcal{I}_R(\Omega) \) where \( \mu \) is the restriction of \( \nu \) to \( A(\Omega_1) \). Thus, there exists an \( \lambda \in X \) such that \( x \lambda < \mu(\Omega_1) \). Now, suppose \( L \in \Omega_2 \) and \( \mu \lambda = 1 \) then \( \lambda \) is on \( L \), \( \nu \lambda = 1 \) and so \( \mu(\Omega_2) = 1 \) for some \( L \) and moreover \( \mu \lambda = 1 \) since \( \mu \lambda < \mu(\Omega_1) \), but \( \mu \lambda = 1 \) then \( \lambda = 1 \) and so \( \Omega_2 \) is weakly compact.

REMARK 5.1. Let \( X \) be a topological space and \( \Theta \) the collection of open sets. Then, by Theorem 5.4, \( \Theta \) is weakly compact iff \( F \) is \( \Theta \)-open almost compact.

Now consider \( X \). Suppose \( \Omega \) is non-disjunctive and define \( \hat{\Omega} = \{ \mu_\lambda: \lambda \in X \} \cup \{ \mu \in \mathcal{I}_R(\Omega): \mu \) is not \( \Omega \)-convergent \} and \( \hat{\mathcal{W}}(\Lambda) = \{ \mu \in \hat{\Omega}: S(\mu) = 1 \} \). We also assume \( \Omega \) is \( T_0 \), so \( \lambda, \gamma \in X \) and \( \lambda \neq \gamma \) implies \( \mu_\lambda \neq \mu_\gamma \).

THEOREM 5.6. For \( \Lambda, \beta \in \mathcal{A}(\Omega) \), we have: a) \( \Lambda = \beta \) iff \( \hat{\mathcal{W}}(\Lambda) = \hat{\mathcal{W}}(\beta) \), b) \( \hat{\mathcal{W}}(\beta) = \hat{\mathcal{W}}(\Lambda) \cup \hat{\mathcal{W}}(\beta) \), c) \( \hat{\mathcal{W}}(\Lambda \cap \beta) = \hat{\mathcal{W}}(\Lambda) \cap \hat{\mathcal{W}}(\beta) \), d) \( \hat{\mathcal{W}}(\Lambda) = \hat{\mathcal{W}}(\Lambda) \), e) \( \hat{\mathcal{W}}(\Lambda) = \hat{\mathcal{W}}(\Lambda) \).

PROOF. a) (i) If \( \Lambda = \beta \), then, clearly \( \hat{\mathcal{W}}(\Lambda) = \hat{\mathcal{W}}(\beta) \). (ii) If \( \Lambda \neq \beta \), then say \( \Lambda \neq \beta \) in \( \mathcal{A}(\Omega) \) then \( \mu_\lambda (A \cap \beta') = 1 \); \( \mu_\beta = 1 \) and so \( \mu_\lambda (A) = 1 \) and \( \mu_\beta (B) = 0 \) which implies that \( \mu \in \hat{\mathcal{W}}(\Lambda) \) and \( \mu \neq \hat{\mathcal{W}}(\beta) \). b), c), d) and e) are not difficult to show and are omitted.

Now consider \( \mu \in \mathcal{I}(\Omega) \) and define \( \hat{\Omega} \) to be \( \mu \)-convergent. Then, one can easily note that \( \mu \in \hat{\Omega} \) 1-1 and onto from \( \mathcal{I}(\Omega) \) to \( \mathcal{I}(\hat{\Omega}) \), and moreover \( \mu \in \mathcal{I}_R(\Omega) \) iff \( \mu \in \mathcal{I}_R(\hat{\Omega}) \).

THEOREM 5.7. \( \hat{\mathcal{W}}(\Omega) \) is weakly compact and \( T_0 \).

PROOF. a) Let \( \mu \in \mathcal{I}_R(\hat{\mathcal{W}}(\Omega)) \) then \( \mu \in \mathcal{I}_R(\Omega) \). If \( \mu \) is \( \Omega \)-convergent then \( \mu_\lambda < \mu(\Omega) \) implies that \( \mu_\lambda < \hat{\mu}(\Omega) \). Note that for \( A \in \mathcal{A}(\Omega) \),

\[
\mu_\lambda (A) = \begin{cases} 1 & \text{if } \lambda \in A \\ 0 & \text{if } \lambda \notin A \end{cases} \quad \text{and } \mu_\lambda \in \hat{\mathcal{W}}(\Lambda) \text{ iff } \mu_\lambda (A) = 1. \text{ Thus, } \mu_\lambda \text{ is the measure concentrated at } \lambda \text{ and so } \hat{\mu} \text{ is } \hat{\mathcal{W}}(\Omega)-\text{convergent. If } \mu \text{ is not } \Omega \text{-convergent then } S(\hat{\mu}) = \hat{\mathcal{W}}(\Lambda) \text{ with } \hat{\mu}(\Lambda) = 1 = \mu(\Lambda), \text{ hence } \mu \in \hat{\mathcal{W}}(\Lambda), \mu \in \mathcal{I}, \mu \in \mathcal{F}(\hat{\mu}) \text{ and } \mu \text{ is the measure concentrated at } \mu, \text{ and consequently } \hat{\mathcal{W}}(\Omega) \text{ is weakly compact. b) Let } \mu_1, \text{ and } \mu_2 \text{,}
\]
There exists an \( L \in \Omega \) such that \( \mu_1(L) = 1, \mu_2(L) = 0 \). Therefore \( \mu_1 \in \mathring{W}(L), \mu_2 \in \mathring{W}(L') \) and so \( \mathring{W}(L) \) is \( T_0 \).

**Theorem 5.8.** If \( \Omega = U(\Omega) \) then \( U(\Omega) \) separates \( U(\mathring{W}(\Omega)) \).

**Proof.** Suppose \( (U \mathring{W}(L_\alpha) \cap (U \mathring{W}(L_\beta)) = \phi \). Let \( A = (U L_\alpha) \in \Omega, B = (U L_\beta) \in \Omega \).

Since \( \Omega = U(\Omega) \) then \( (U \mathring{W}(L_\alpha)) \subseteq W(A) \) and \( (U \mathring{W}(L_\beta)) \subseteq W(B) \). If \( \alpha \cap \beta = \phi \) then \( \mathring{W}(L_\alpha) \cap \mathring{W}(L_\beta) \neq \phi \) for some \( \alpha, \beta \). Let \( x \in L_\alpha \cap L_\beta \) then \( x \in \mathring{W}(L_\alpha) \) and \( x \in \mathring{W}(L_\beta) \) which contradicts (1). Thus, \( A \cap B = \phi \), \( \mathring{W}(A) \cap \mathring{W}(B) = \phi \), and the desired result is now clear.

Now, we note that if \( \Omega = U(\Omega) \) then \( \Omega \) with \( \mathring{W}(\Omega) \) is generalized absolutely closed and is absolutely closed if \( \Omega \) is \( T_2 \). Thus, if we consider \( X \) and let \( \Omega = U(\Omega) = \emptyset \) and \( T_2 \), then \( \Omega \) is an absolute closure of \( X \) since one can easily observe that \( X = \mathring{W}(X) \).

**Remark.** An analogous construction can now be done for \( \Omega^\sigma, \mathring{W}(\Omega) \). Where \( \Omega^\sigma = \{ x : x \in X \} \cup \{ \mu \in I^\sigma(\Omega) : \mu \) is not \( \Omega \)-convergent \} and \( \mathring{W}(\Omega) = \{ \mu \in I^\sigma, \mu(A) = 1, A \in A(\Omega) \} \) and one can show that \( \mathring{W}(\Omega) \) is weakly replete. \( \Omega \) is weakly replete if for any \( \mu \in I^\sigma(\Omega), \mu \) is \( \Omega \)-convergent. We note that the constructions here generalize the work of Lui [6].

**References**


3. BACHMAN, G. and STRATIGOS, P., Criteria for \( \sigma \)-smoothness, \( \tau \)-smoothness, and tightness of Lattice Regular Measures, with Applications, Con. J. Math. 33(1981), 1498-1525.


