NOTE ON CERTAIN SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS OF ORDER $\alpha$

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ABSTRACT. The object of the present paper is to show a result for functions belonging to the class $R(\alpha)$ which is the subclass of close-to-convex functions in the unit disk $U$.

KEY WORDS AND PHRASES. Close-to-convex of order $\alpha$, class $P(\alpha)$, class $R(\alpha)$, starlikeness bound.

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1. INTRODUCTION.

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disk $U = \{z: |z| < 1\}$. A function $f(z)$ belonging to the class $A$ is said to be close-to-convex of order $\alpha$ if and only if it satisfies the condition

$$\text{Re}[f'(z)] > \alpha$$

for some $\alpha (0 < \alpha < 1)$ and for $z \in U$. We denote by $P(\alpha)$ the subclass of $A$ consisting of functions which are close-to-convex of order $\alpha$ in the unit disk $U$. 
Further, let \( R(\alpha) \) be the subclass of \( A \) consisting of all functions which satisfy the condition

\[
|f'(z) - 1| < 1 - \alpha
\]

for some \( \alpha (0 < \alpha < 1) \) and for all \( z \in U \).

It is clear that \( R(\alpha) \subset P(\alpha) \) for \( 0 < \alpha < 1 \). Nunokawa, Fukui, Owa, Saitoh and Sekking [1] have showed the starlikeness bound of functions in the class \( R(\alpha) \). Also, the starlikeness bound of functions belonging to the class \( P(\alpha) \) was given by Fukui, Owa, Ogawa and Nunokawa [2].

2. MAIN RESULT.

In order to prove our main result, we have to recall here the following lemma due to Lewandowski, Miller and Ziotkiewicz [3].

**LEMMA.** Let \( \beta \) be real and \( |\beta| < \pi/2 \). Let \( \phi(u,v) \) be a complex valued function \( \phi: D \to C \), \( D \subset C \times C \) (\( C \) is the complex plane), and let \( u = u_1 + iu_2, v = v_1 + iv_2 \). Suppose that the function \( \phi(u,v) \) satisfies

(i) \( \phi(u,v) \) is continuous in \( D \);

(ii) \( (e^{i\beta},0) \in D \) and \( \text{Re}\{\phi(e^{i\beta},0)\} > 0 \);

(iii) \( \text{Re}\{\phi(iu_2,v_1)\} < 0 \) when \( (iu_2,v_1) \in D \) and

\[
\frac{1 - 2u_2\sin\beta + u_2^2}{2\cos\beta} v_1 < 0
\]

Let \( p(z) = e^{i\beta} + p_1z + p_2z^2 + \ldots \) be regular in the unit disk \( U \) such that \( (p(z),zp'(z)) \in D \) for all \( z \in U \). If

\[
\text{Re}\{\phi(p(z),zp'(z))\} > 0 \quad (z \in U),
\]

then \( \text{Re}\{p(z)\} > 0 \) \( (z \in U) \).

Applying the above lemma, we derive

**THEOREM.** Let the function \( f(z) \) defined by (1.1) be in the class \( R(\alpha) \). Then

\[
\text{Re}\left( e^{i\beta} \frac{f(z)}{z} \right) > 0,
\]
where

\[ \left| \beta \right| < \frac{\pi}{2} - \sin^{-1}(1 - \alpha). \quad (2.2) \]

**Proof.** It follows from \( f(z) \in R(\alpha) \) that

\[ \Re\{e^{i\beta}f(z)\} > 0 \quad (z \in U) \quad (2.3) \]

for \( \left| \beta \right| < \pi/2 - \sin^{-1}(1 - \alpha) \). Defining the function \( p(z) \) by

\[ e^{i\beta} \frac{f(z)}{z} = p(z), \quad (2.4) \]

we can see that \( p(z) = e^{i\beta} + p_1z + p_2z^2 + \ldots \) is regular in \( U \). Taking the differentiations of both sides in (2.4), we have

\[ e^{i\beta}f'(z) = p(z) + zp'(z). \quad (2.5) \]

It follows from (2.3) that

\[ \Re\{e^{i\beta}f'(z)\} = \Re\{p(z) + zp'(z)\} > 0. \quad (2.6) \]

Setting

\[ \phi(u,v) = u + v \quad (\text{note that } u = p(z) \text{ and } v = zp'(z)), \quad (2.7) \]

we see that

(i) \( \phi(u,v) \) is continuous in \( D = C \times C \);

(ii) \( (e^{i\beta}, 0) \in D \) and \( \Re\{\phi(e^{i\beta}, 0)\} = \cos \beta > 0 \);

(iii) for all \( (iu_2, v_1) \in D \) such that

\[ v_1 < \frac{1 - 2u_2\sin \beta + u_2^2}{2\cos \beta}, \]

\[ \Re\{\phi(iu_2, v_1)\} = v_1 \]

\[ \leq \frac{1 - 2u_2\sin \beta + u_2^2}{2\cos \beta} < 0, \]

Therefore, the function \( \phi(u,v) \) defined by (2.7) satisfies the conditions in Lemma.
Using Lemma, we have

\[ \text{Re}\{\mu(z)\} = \text{Re}\{e^{\frac{18}{z}} f(z)\} > 0 \]

which completes the proof of Theorem.

Letting \( \alpha = 0 \) in Theorem, we have

**COROLLARY.** Let the function \( f(z) \) defined by (1.1) be in the class \( R(0) \). Then

\[ \text{Re}\left\{ \frac{f(z)}{z}\right\} > 0 \quad (z \in U). \]

**REFERENCES**

